Non-Abelian Confinement via Abelian Flux Tubes in Softly Broken $\mathcal{N} = 2$ SUSY QCD

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Abstract

We study confinement in softly broken $\mathcal{N} = 2$ SUSY QCD with gauge group $SU(N_c)$ and N_f hypermultiplets of fundamental matter (quarks) when the Coulomb branch is lifted by small mass of adjoint matter. Concentrating mostly on the theory with $SU(3)$ gauge group we discuss the $\mathcal{N} = 1$ vacua which arise in the weak coupling at large values of quark masses and study flux tubes and monopole confinement in these vacua. In particularly we find the BPS strings in $SU(3)$ gauge theory formed by two interacting $U(1)$ gauge fields and two scalar fields generalizing ordinary Abrikosov-Nielsen-Olesen vortices. Then we focus on the $SU(3)$ gauge theories with $N_f = 4$ and $N_f = 5$ flavors with equal masses. In these theories there are $\mathcal{N} = 1$ vacua with restored $SU(2)$ gauge subgroup in quantum theory since $SU(2)$ subsectors are not asymptotically free. We show that although the confinement in these theories is due to Abelian flux tubes the multiplicity of meson spectrum is the same as expected in a theory with non-Abelian confinement.

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According to Mandelstam, Polyakov and 't Hooft [1] confinement of charges arises as Meissner effect upon condensation of monopoles. Once monopoles condense the electric flux is confined within the electric flux tube [2, 3] connecting the heavy trial charge and anti-charge. The flux tube has constant energy per unit length – the string tension $T = (2\pi\alpha')^{-1}$, and this ensures that confining potential between heavy charge and anti-charge increases linearly with their separation. However, since dynamics of monopoles is hard to control in non-supersymmetric gauge theories, this picture of confinement for many years remained to be an unjustified qualitative scheme.

The breakthrough in this direction was made by Seiberg and Witten in [4, 5]. Constructing exact solution to $\mathcal{N}=2$ supersymmetric gauge theory they have shown that the condensation of monopoles really occurs near the monopole point in the moduli space of the theory once $\mathcal{N}=2$ supersymmetry is broken down to $\mathcal{N}=1$ by the mass term of adjoint matter [4]. After the work of Seiberg and Witten it has become very important to understand to what extent this Abelian confinement of electric charges is similar to confinement of color we expect (but cannot control) in real QCD. Moreover, one expects the QCD-like confinement also in $\mathcal{N}=1$ supersymmetric QCD which can be obtained as a limit of large mass of the adjoint matter μ of softly broken $\mathcal{N}=2$ QCD, but again, we have no control on the dynamics of this theory in the large μ limit.

One important distinction noticed by Douglas and Shenker [6] appears in $SU(N_c)$ gauge theories with $N_c \geq 3$. Since $SU(N_c)$ gauge group is broken down to $U(1)^{N_c-1}$ by the VEV's of adjoint scalars there are $N_c - 1$ generically different flux tubes, one per each $U(1)$ factor. Numerous flux tubes lead to existence of too many hadronic states in the spectrum [6] (see also [7] for the D-brane reinterpretation of this result). In particularly, there are N_c different sets of quark-antiquark meson Regge trajectories [1]. For example, in pure $SU(N_c)$ gauge theory the number of families of sets of these trajectories with different slope is the integer part of $(N_c+1)/2$. The presence of many quark-antiquark meson trajectories reflects the essentially Abelian nature of confinement in Seiberg-Witten theory.

On the other hand in $\mathcal{N}=1$ supersymmetric theories described in the framework of Seiberg's duality [8] we should get non-Abelian confinement since without adjoint fields breaking of the gauge symmetry down to Abelian subgroup does not occur. It is believed that condensation of "magnetic quarks" of non-Abelian dual theory leads to confinement of ordinary quarks. Still the mechanism of this confinement is not understood. The problem is that usually non-Abelian dual gauge groups, say $SU(r)$ groups do not admit flux tubes.

The bridge between two approaches was suggested in [9, 10]. It was shown that certain $\mathcal{N}=1$ vacua of softly broken $\mathcal{N}=2$ QCD at small μ preserve non-Abelian $SU(r)$ subgroups of $SU(N)$ gauge symmetry ($r < N_c$). The theories at some of these vacua correspond at large μ to $\mathcal{N}=1$ QCD described by Seiberg's duality. This suggests that confinement at these vacua should be non-Abelian even at small μ. In particularly we should have only one set of quark-antiquark meson Regge trajectories. In this paper we are going to study the outlined above proposal for the non-Abelian confinement from the side of softly broken $\mathcal{N}=2$ QCD at small μ. In particular we analyse what happens to Abelian flux tubes when a non-Abelian subgroup of gauge symmetry is restored at certain $\mathcal{N}=1$ vacua. We will mostly focus on theory with the $SU(3)$ gauge group and $N_f \leq 5$ flavors of fundamental hypermultiplets to be called as quarks below. In sect. 2 we review the vacuum structure of the theory studied partially in [9, 10]. We consider the case of large quark masses m_A, $A = 1, \dots N_f$, to keep theory at weak coupling and discuss vacua where quarks develop VEV's. These vacua are classified by the number r of different colors for which quarks have non-zero VEV's. For $SU(3)$ gauge theory the isolated vacua can have $r = 0, 1, 2$ and we are interested mostly in weak coupling vacua with $r = 1$ and/or $r = 2$ for large values of m_A.

In sect. 3 we present weak coupling Abelian description of the low energy effective theory around these vacua and discuss the low energy spectra. Generically the $SU(3)$ gauge group is broken down to its maximal $U(1) \times U(1)$ Abelian subgroup by the VEV's of adjoint matter and we have two light photon multiplets. In sect. 4 and sect. 5 we study in detail the flux tubes in $r = 1$ and $r = 2$ vacua. It turns out that $r = 1$ vacua possess the standard Abrikosov-Nielsen-Olesen (ANO) flux tubes [2, 3] described by one $U(1)$ gauge field and one scalar field. In the limit of small masses of the adjoint matter μ these flux tubes are BPS-saturated [11]. However, at $r = 2$ vacua the flux tubes turn out to have more complicated structure. They are formed now by two gauge fields interacting with two scalar fields. We consider the lattice of these flux tubes of more general type and study which ones among them are BPS states. To do this we calculate the interaction potential of strings at large distances.

In sect. 6 we consider finally the $r = 2$ vacua in the $SU(3)$ gauge theories with $N_f = 4$ and $N_f = 5$

[1]Each set corresponds to main trajectory together with "daughter" trajectories.

reason is that classically restored $SU(2)$ subgroup stays unbroken also at quantum level since corresponding $SU(2)$ subsectors are not asymptotically free (though, of course, the $SU(3)$ gauge theory with $N_f = 4$ and $N_f = 5$ is asymptotically free itself). We study what happens to the Abelian flux tubes upon restoration of the $SU(2)$ gauge subgroup and find that one of two "elementary" flux tubes becomes unstable and eventually disappears from the spectrum. The monopole confinement is now due to the only one remaining flux tube and this eliminates unwanted multiplicity of the hadron spectrum mentioned above. In fact, in this vacuum we have only one set of meson Regge trajectories as expected in a theory with non-Abelian confinement.

2 Vacuum structure

2.1 Superpotential and vacuum equations

Consider softly broken $\mathcal{N} = 2$ $SU(N_c)$ gauge theory with N_f flavors. The field content in terms of $\mathcal{N} = 1$ supermultiplets can be described by the gauge multiplet W_α and chiral multiplet Φ in the adjoint representation (forming together $\mathcal{N} = 2$ gauge supermultiplet) so that first contains the gauge field A_μ^{ij} and complex Weyl fermion λ_α^{ij}, while the second – complex scalar Φ^{ij} and complex Weyl fermion ψ_α^{ij}, $(i,j = 1,\ldots,N_c)$, – all being $N_c \times N_c$ matrices with zero trace and $2N_f$ chiral multiplets of matter Q_A^i and $\tilde{Q}_A^i$ in the N_c and $\bar{N}_c$ representations respectively, i.e. $i = 1,\ldots,N_c$ and $A = 1,\ldots,N_f$. The vacuum structure we are going to discuss can be associated with extrema $\delta\mathcal{W}$ of the superpotential

$$\mathcal{W} = \sum_{A=1}^{N_f}\left(\sqrt{2}\tilde{Q}_{A,i}\Phi_{ij}Q_{A,j} + m_A\tilde{Q}_{A,i}Q_{A,i}\right) + \sum_{k=2}^{N_c-1}\mu_k \mathrm{Tr}\Phi^k \tag{1}$$

together with vanishing of the D-terms $[\Phi^\dagger,\Phi] = 0$ and

$$D_{ij} \equiv \sum_{A=1}^{N_f}\left(Q_{A,i}\bar{Q}_{A,j} - \tilde{Q}_{A,i}\bar{\tilde{Q}}_{A,j}\right) = \nu\delta_{ij} \tag{2}$$

The parameter $\nu = \nu(Q,\tilde{Q},\bar{Q},\bar{\tilde{Q}})$ is in fact some function of quark's VEV's and it can be nonzero only if *all* $Q_j \neq 0$ for $j = 1,\ldots,N_c$. The variation of the superpotential (1) gives rise to the following set of equations

$$\begin{aligned}\sqrt{2}\Phi_{ij}Q_{A,j} + m_A Q_{A,i} = 0 \quad &\forall\ A = 1,\ldots,N_f\\ \sqrt{2}\tilde{Q}_{A,i}\Phi_{ij} + m_A\tilde{Q}_{A,j} = 0 \quad &\forall\ A = 1,\ldots,N_f\end{aligned} \tag{3}$$

including also F-term condition $\frac{\partial\mathcal{W}}{\partial\Phi_{ij}} = 0$, which should be taken into account together with (2). One should also remember that $\mathrm{Tr}\Phi = 0$, say, introducing the Lagrange multiplier μ_1 into (1), the same is to impose vanishing condition only onto the traceless part

$$\sqrt{2}\sum_{A=1}^{N_f}\left(Q_{A,i}\tilde{Q}_{A,j} - \frac{\delta_{ij}}{N_c}\left(\sum_k Q_A^k\tilde{Q}_{Ak}\right)\right) + \sum_{k\geq 2}k\mu_k\left(\left(\Phi^{k-1}\right)_{ji} - \frac{\delta_{ij}}{N_c}\mathrm{Tr}\Phi^{k-1}\right) = 0 \tag{4}$$

of $\frac{\partial\mathcal{W}}{\partial\Phi_{ij}} = \sqrt{2}F_{ij} + \sum_{k\geq 2}k\mu_k\left(\Phi^{k-1}\right)_{ji}$ with

$$F_{ij} \equiv \sum_{A=1}^{N_f}Q_{A,i}\tilde{Q}_{A,j} \tag{5}$$

Let us first investigate the structure of solutions to vacuum equations (2), (3) and (4). We shall basically use quasiclassical regime when all masses of the quarks $m_A \gg \Lambda_{QCD}$ where the coupling is small and quasiclassics is a good approximation. Whenever it is not possible we will add extra arguments based on the Seiberg-Witten analysis of strong coupling regime of SUSY gauge theories [4, 5].

Equation $[\Phi^\dagger,\Phi] = 0$ gives rise immediately to the conclusion that, in general position, up to gauge trans-

$$
\Phi = \begin{pmatrix} \phi_1 & 0 & \dots & 0 \\ 0 & \phi_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \phi_{N_c} \end{pmatrix}
$$
$$
\mathrm{Tr}\Phi = \sum_{i=1}^{N_c} \phi_i = 0 \tag{6}
$$

and gauge group $SU(N_c)$ is broken down to $U(1)^{N_c-1}$ since the off-diagonal vector fields A_μ^{ij} acquire the masses proportional to

$$
[\Phi, A_\mu]_{ij} = (\phi_i - \phi_j)A_\mu^{ij} \tag{7}
$$

due to the Higgs effect. When $\phi_i = \phi_j$ for some $i \neq j$ non-Abelian gauge symmetry may be partially restored. For diagonal Φ (6) the second term in (4) vanishes for $i \neq j$ and one ends up with a simple problem of counting of the eigenvectors of matrix Φ. We shall consider in detail the case of gauge group $SU(3)$.

2.2 SU(3) gauge group

The $SU(3)$ $\mathcal{N} = 1$ SUSY gauge theory without matter has exactly $N_c = 3$ vacua, all in the strong coupling regime. This is a particular case of general situation for an $SU(N_c)$ pure gauge theory, which has N_c points in the moduli space when $N_c - 1$ monopoles become massless (see, for example, [12, 13, 6]).

Let us add one flavor with mass $m_A = m$ so that the equations (3) and linear combinations of the equations (4) turn into

$$
\begin{aligned}
(\sqrt{2}\phi_1 + m)Q_1 = 0 \qquad & \tilde{Q}_1(\sqrt{2}\phi_1 + m) = 0 \\
(\sqrt{2}\phi_2 + m)Q_2 = 0 \qquad & \tilde{Q}_2(\sqrt{2}\phi_2 + m) = 0 \\
(-\sqrt{2}\phi_1 - \sqrt{2}\phi_2 + m)Q_3 = 0 \qquad & \tilde{Q}_3(-\sqrt{2}\phi_1 - \sqrt{2}\phi_2 + m) = 0 \\
\sqrt{2}\left(\tilde{Q}_1 Q_1 - \tilde{Q}_3 Q_3\right) &= -(2\mu_2 - 3\mu_3\phi_2)(2\phi_1 + \phi_2) \\
\sqrt{2}\left(\tilde{Q}_2 Q_2 - \tilde{Q}_3 Q_3\right) &= -(2\mu_2 - 3\mu_3\phi_1)(\phi_1 + 2\phi_2)
\end{aligned} \tag{8}
$$

where we have restricted ourselves to (the only essential in $SU(3)$ case) nonzero μ_2 and μ_3. Matrix Φ may have only one eigenvector, which can be by gauge transformation turned to any given direction in colour space, for example [2]

$$
Q_{A,i} \equiv Q_i = u\delta_{i1} = u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{9}
$$

and

$$
\tilde{Q}_{A,i} \equiv \tilde{Q}_i = \tilde{u}\delta_{i1} = \tilde{u} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{10}
$$

Taking into account (9), (10) equations (8) give rise to $\sqrt{2}\phi_1 + m = 0$ or $\phi_1 = -\frac{m}{\sqrt{2}}$. Then, from the last equation of (8) and vanishing of Q_2 and Q_3 one gets that $\phi_2 = -\frac{\phi_1}{2} = \frac{m}{2\sqrt{2}}$, and, as a consequence $\phi_3 = -\phi_1 - \phi_2 = \frac{m}{2\sqrt{2}}$. As a result matrix Φ acquires the form

$$
\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -m & 0 & 0 \\ 0 & \frac{m}{2} & 0 \\ 0 & 0 & \frac{m}{2} \end{pmatrix} \tag{11}
$$

It means that the gauge group is in fact broken only up to $SU(2) \times U(1)$. The vector fields A_μ^{12} and A_μ^{13} (and their superpartners) become very heavy, of the mass $\sim m \gg \Lambda$, while the mass of the photon A_μ^{11} and its

[2] In the case of the $SU(3)$ gauge group we will use common notations u, d and s for the Q_i with $i = 1, 2, 3$.

$$m^{\rm light} \sim g\sqrt{m\left(\mu_2 - \frac{3}{4\sqrt{2}}\mu_3 m\right)} \underset{\mu_3=0}{\equiv} g\sqrt{m\mu} \tag{12}$$

i.e. is of the scale of SUSY breaking for generic μ_2 and μ_3 [3]. In (12) we have also introduced the coupling constant g which comes from kinetic term and corresponds to correct normalization of mass as a pole in the propagator. From the equations (8) and formulas (9) and (10) we immediately find the value of the quark condensate $\langle\tilde{Q}Q\rangle$ in this vacuum

$$\begin{aligned} \langle\tilde{u}u\rangle &= \frac{3}{2}\,\mu m \equiv \frac{\zeta}{2} \\ \langle\tilde{d}d\rangle &= \langle\tilde{s}s\rangle = 0 \end{aligned} \tag{13}$$

The $SU(2)$ subgroup is unbroken at the scale of order of m, it interacts only with very heavy matter Q_2 and Q_3 (of $m \gg \Lambda$) and by the Seiberg-Witten mechanism it runs to the strong-coupled phase and has two Seiberg-Witten vacua [4, 5]. It means that vacua of the full theory would correspond to

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} -m & 0 & 0 \\ 0 & \frac{m}{2} \pm \Lambda_{SU(2)} & 0 \\ 0 & 0 & \frac{m}{2} \mp \Lambda_{SU(2)} \end{pmatrix} \tag{14}$$

where $\Lambda^4_{SU(2)}m = \Lambda^5_{SU(3)} \equiv \Lambda^5$. The gauge fields A^{23}_μ and their superpartners would get masses of the order of $\Lambda_{SU(2)} \ll m$ and the unbroken $U(1)$ subgroup of this $SU(2)$ will remain light, with the mass $\sim \sqrt{\mu\Lambda_{SU(2)}}$. This mass is even less than (12). In order to avoid running of the $SU(2)$ subgroup into the strong coupling one has to add more flavors to our gauge theory.

2.3 More flavors with SU(3)

Let us now turn to the situation with more flavors. Of course we can still have vacua of the type considered in previous section for each flavor, however, in this case more complicated vacua when different flavors get simultaneously non-zero VEV's also arise.

From (3) it follows that either Q_A and $\tilde{Q}_A$ are zero for a given A or they are eigenvectors of the diagonal matrix (6) with eigenvalue $-m_A/\sqrt{2}$. An important restriction on extra vacua also comes from the formula (4) with $i \neq j$ (see (6))

$$F_{ij} = \sum_{A=1}^{N_f} Q_{A,i}\tilde{Q}_{A,j} = 0, \qquad i \neq j \tag{15}$$

Combining these conditions together we see that for example, in the case $N_f = 2$ one may consider only $Q_{A,i} = Q_i\delta_{Ai}$, i.e.

$$\begin{aligned} Q_{1,i} &= u_1\delta_{i1} = u_1\begin{pmatrix}1\\0\\0\end{pmatrix} \\ Q_{2,i} &= d_2\delta_{i2} = d_2\begin{pmatrix}0\\1\\0\end{pmatrix} \end{aligned} \tag{16}$$

and $\tilde{Q}_{A,i} = \tilde{Q}_i\delta_{Ai}$, or

$$\begin{aligned} \tilde{Q}_{1,i} &= \tilde{u}_1\delta_{i1} = \tilde{u}_1\begin{pmatrix}1 & 0 & 0\end{pmatrix} \\ \tilde{Q}_{2,i} &= \tilde{d}_2\delta_{i2} = \tilde{d}_2\begin{pmatrix}0 & 1 & 0\end{pmatrix} \end{aligned} \tag{17}$$

[3]In this paper we do not consider possible complications related to the "fine-tuning" of (12), say when for $\mu_3 = \frac{4\sqrt{2}\mu_2}{3m}$ extra fields could become massless, at least classically.

$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & -m_1 - m_2 \end{pmatrix} \tag{18}$$

This is a general situation up to gauge rotations in the colour space, so one gets an extra vacuum with two $U(1)$ gauge groups softly broken at "light" level of (12). Indeed, from (16), (17) and, following from (4) relations one finds the following values for the vacuum condensates:

$$\begin{aligned} \langle \tilde{u}_1 u_1 \rangle &= -\frac{1}{\sqrt{2}}(2\mu_2 - 3\mu_3\phi_2)(2\phi_1 + \phi_2) = (\mu_2 + \frac{3}{2\sqrt{2}}\mu_3 m_2)(2m_1 + m_2) \underset{\mu_3=0}{\equiv} \mu(2m_1 + m_2) \\ \langle \tilde{d}_2 d_2 \rangle &= -\frac{1}{\sqrt{2}}(2\mu_2 - 3\mu_3\phi_1)(\phi_1 + 2\phi_2) = (\mu_2 + \frac{3}{2\sqrt{2}}\mu_3 m_1)(m_1 + 2m_2) \underset{\mu_3=0}{\equiv} \mu(m_1 + 2m_2) \end{aligned} \tag{19}$$

The spectrum of light fields will be discussed in detail in sect. 3.3. The total number of vacua is in agreement with the formula from [10]

$$\#(\text{vacua}) = \sum_{r=0}^{\min(N_c-1, N_f)} (N_c - r) C_r^{N_f} \tag{20}$$

where r counts the number of nontrivial eigenvectors or solutions to (3). Indeed, $r = 0$ term gives N_c ($= 3$ for $SU(3)$) Seiberg-Witten vacua "without matter", $r = 1$ term adds $(N_c - 1) \cdot N_f$ ($= 2N_f$ for $SU(3)$) vacua, corresponding to (14) for each flavor – with the gauge group broken to $SU(N_c - 1)$ at the scale $m \gg \Lambda$ in weak coupling. The term with $r = 2$

$$(N_c - 2) C_2^{N_f} \underset{N_c=3}{=} \frac{N_f(N_f - 1)}{2} \tag{21}$$

corresponds exactly to the situation, considered in this section.

Formula (20) has a simple physical meaning. The factor $(N_c - r)$ is exactly the Witten index of unbroken by the adjoint matter gauge group (in the case we consider adjoint matter always breaks $SU(N_c)$ down to some "lower" $SU(N_c - r)$). The combinatorial factor $C_r^{N_f}$ counts the number of possibilities to arrange quark VEV's within N_f flavors.

2.4 Colliding vacua and Higgs branches

If some of the masses of the matter multiplets coincide ($m_A = m_B$ for $A \neq B$; $A, B = 1, \ldots, N_f$) one gets a Higgs branch with VEV's $\langle Q \rangle \neq 0$, "growing" from the corresponding point on the moduli space of the Coulomb branch. The (real) dimension of the Higgs branch is $4\mathcal{H}$, i.e. the dimension of the hyper Kähler manifold is four times the "number of hypermultiplets" $\mathcal{H}$.

In the $r = 1$ case one can always choose the nontrivial eigenvectors of Φ along some fixed direction in the colour space, say only $Q^{1A} \neq 0$ and $\tilde{Q}_{A1} \neq 0$ for any A. Then we get $4N_f$ real parameters (for coinciding all $m_A = m$ and matrix Φ of the form of (11)), which should obey two real F-term and one real D-term relations. One extra degree is "eaten up" by the $U(1)$ gauge group so that finally the dimension of the Higgs branch appears to be $4N_f - 3 - 1 = 4(N_f - 1)$ or $\mathcal{H}_{r=1} = N_f - 1$.

In the $r = 2$ case the situation is a bit more complicated. The solutions for nontrivial Q and $\tilde{Q}$ can be now chosen in the form of the following rectangle matrices

$$\|Q^{kA}\| = \begin{pmatrix} Q^{11} & Q^{12} & \ldots & Q^{1N_f} \\ Q^{21} & Q^{22} & \ldots & Q^{2N_f} \\ 0 & 0 & \ldots & 0 \end{pmatrix} \tag{22}$$

and

$$\|\tilde{Q}_{Ak}\| = \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & 0 \\ \tilde{Q}_{21} & \tilde{Q}_{22} & 0 \\ \vdots & \vdots & \vdots \\ \tilde{Q}_{N_f1} & \tilde{Q}_{N_f2} & 0 \end{pmatrix} \tag{23}$$

four real D-term conditions (2) and eight real F-term conditions and gauge group $SU(2)\times U(1)$ "eats" four "phases", so that the total number of independent parameters is $8N_f-12-4=8(N_f-2)$ or $\mathcal{H}_{r=2}=2(N_f-2)$, which coincides with the formula of [9] for $r=2$.

2.5 Baryonic branches

Formula (20) is based on counting of the number r of nonzero eigenvectors – solutions to (3). We have considered the cases $r=1$ and $r=2$ for the $SU(3)$ gauge theory, but it is obvious that the maximal number of eigenvectors is $r=N_c=3$. This possibility appears first time for $N_f=N_c$ ($=3$ if we still consider the $SU(3)$ gauge group) and is only possible, however, in the situation when $\sum_{A=1}^{N_f}m_A=\sum_{A=1}^{3}m_A=0$.

Consider, for example, $r=2$ vacuum of the previous section. The relation (15) still requires $Q_{A,i}=Q_i\delta_{Ai}$ and $\tilde{Q}_{A,i}=\tilde{Q}_i\delta_{Ai}$, but now for $i,A=1,\ldots,3$. In this way we get the set of vacua, parameterized by

$$\begin{aligned}\sum_{A=1,2}\left(\tilde{Q}_{A,1}Q_{A,1}-\tilde{Q}_{A,3}Q_{A,3}\right)&=\tilde{u}_1u_1-\tilde{s}_3s_3=(\mu_2+\frac{3}{2\sqrt{2}}\mu_3m_2)(2m_1+m_2)\\ \sum_{A=1,2}\left(\tilde{Q}_{A,2}Q_{A,2}-\tilde{Q}_{A,3}Q_{A,3}\right)&=\tilde{d}_2d_2-\tilde{s}_3s_3=(\mu_2+\frac{3}{2\sqrt{2}}\mu_3m_1)(m_1+2m_2)\end{aligned}\tag{24}$$

The system (24) describes the baryonic branch [4], indeed (19) is simply obtained from (24) putting $Q_3=\tilde{Q}_3=0$, and in this point m_3 can be "unfrozen" from $-(m_1+m_2)$. This is a baryonic branch since the VEV's of baryons

$$\begin{aligned}B&=u_1d_2s_3=\frac{1}{3!}\epsilon_{ijk}\epsilon_{ABC}Q_{A,i}Q_{B,j}Q_{C,k}\\ \tilde{B}&=\tilde{u}_1\tilde{d}_2\tilde{s}_3=\frac{1}{3!}\epsilon_{ijk}\epsilon_{ABC}\tilde{Q}_{A,i}\tilde{Q}_{B,j}\tilde{Q}_{C,k}\end{aligned}\tag{26}$$

are nonzero. The dimension of this baryonic branch is $\#(Q_i,\tilde{Q}_i)-\#(F)-\#(D)-\#(\text{phases})=12-4-2-2=4$.

The simplest analog of this branch exists already for the $SU(2)$ gauge theory with $N_f=N_c=2$ matter hypermultiplets with $m_1=-m_2=m$. The corresponding moduli space is described by

$$\Phi=\frac{1}{\sqrt{2}}\begin{pmatrix}-m&0\\0&m\end{pmatrix}\tag{27}$$

and

$$\begin{aligned}Q^{kA}&=\begin{pmatrix}Q^1&0\\0&Q^2\end{pmatrix}\\ \tilde{Q}_{Ak}&=\begin{pmatrix}\tilde{Q}_1&0\\0&\tilde{Q}_2\end{pmatrix}\end{aligned}\tag{28}$$

or eight real parameters and the relations (2) and (4) give one and two real relations on them correspondingly (the last one

$$Q^1\tilde{Q}_1-Q^2\tilde{Q}_2=2\mu m\tag{29}$$

is a particular case of counted carefully general situation when the $i=j$ relations of (4) impose $2\cdot\text{rank}=2\cdot(N_c-1)$ conditions) plus a $U(1)$ phase, so that the dimension is $8-2-1-1=4$. The corresponding baryon operators are $B=Q^1Q^2$ and $\tilde{B}=\tilde{Q}_1\tilde{Q}_2$.

One may also add more flavors, up to $N_f=4$ (the conformal point for the $SU(2)$ theory) when the baryonic

[4] To "symmetrize" (24) one may also add a relation

$$\sum_{A=1,2}\left(\tilde{Q}_{A,1}Q_{A,1}-\tilde{Q}_{A,2}Q_{A,2}\right)=\tilde{u}_1u_1-\tilde{d}_2d_2=(\mu_2+\frac{3}{2\sqrt{2}}\mu_3m_3)(m_1-m_2)\tag{25}$$

which is not independent, but is just the difference of two in (24).

$$
Q^{kA} = \begin{pmatrix} Q^1 & 0 & Q^3 & 0 \\ 0 & Q^2 & 0 & Q^4 \end{pmatrix}
$$

$$
\tilde{Q}_{Ak} = \begin{pmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_2 \\ \tilde{Q}_3 & 0 \\ 0 & \tilde{Q}_4 \end{pmatrix} \tag{30}
$$

modulo (2) and (4) which gives $16 - 2 - 1 - 1 = 12$ parameters or three hypermultiplets.

3 Low energy mass spectrum

In this section we work out the action of the effective low energy Abelian theory and use it to study the low energy spectrum in isolated $\mathcal{N} = 1$ charge vacua of the theory at $m_A \gg \Lambda$ (i.e. at different values of quark masses when there are no Higgs branches). We put $\mu_k = \mu\delta_{k2}$ or keep only $\mu_2 \equiv \mu \neq 0$, assuming also that $\mu \ll \Lambda$.

3.1 Abelian description

Let us consider scales of order $\sqrt{\mu m_A}$ which are well below W-boson masses at small μ. At this scale $SU(3)$ gauge group is broken down to $U(1)^2$ by VEV of the adjoint scalar (6)

$$
\Phi = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a_3 + \frac{a_8}{\sqrt{3}} & 0 & 0 \\ 0 & -a_3 + \frac{a_8}{\sqrt{3}} & 0 \\ 0 & 0 & -2\frac{a_8}{\sqrt{3}} \end{pmatrix} \equiv \lambda_3 a_3 + \lambda_8 a_8 \tag{31}
$$

at generic values at quark masses. We also have two $U(1)$ gauge fields (in the orthogonal basis denoted $A_\mu^{(3)}$ and $A_\mu^{(8)}$; note that the orthogonal basis is normalized so that $\mathrm{Tr}\Phi^2 = \frac{1}{2}\left(a_3^2 + a_8^2\right)$) introduced via

$$
A_\mu = \frac{1}{2} \begin{pmatrix} A_\mu^{(3)} + \frac{A_\mu^{(8)}}{\sqrt{3}} & 0 & 0 \\ 0 & -A_\mu^{(3)} + \frac{A_\mu^{(8)}}{\sqrt{3}} & 0 \\ 0 & 0 & -2\frac{A_\mu^{(8)}}{\sqrt{3}} \end{pmatrix} = \lambda_3 A_\mu^{(3)} + \lambda_8 A_\mu^{(8)} \tag{32}
$$

where our notations correspond to expanding (flavor) gauge fields either in the basis given by (divided by two) root vectors in the Cartan subalgebra of $SU(3)$ $\boldsymbol{\alpha}_{12}/2$ and $\boldsymbol{\alpha}_2/2$ [5], see Appendix and fig. 1, or, what is more adequate for study explicit solutions, in the orthogonal basis of the Gell-Mann matrices – the basis of diagonal Cartan generators of the $SU(3)$ Lie algebra (see (148) and (149) in Appendix). In these notations the bosonic part of the low energy effective Abelian theory acquires the form

$$
\begin{aligned}
S_{QED} = \int d^4x \Bigg(& \frac{1}{4g^2}\left(F_{\mu\nu}^{(3)}\right)^2 + \frac{1}{4g^2}\left(F_{\mu\nu}^{(8)}\right)^2 + \frac{1}{g^2}\left|\partial_\mu a_3\right|^2 + \frac{1}{g^2}\left|\partial_\mu a_8\right|^2 + \\
& + \sum_{i=u,d,s}\left(\left|\nabla_\mu^{(i)} Q_i^A\right|^2 + \left|\nabla_\mu^{(i)} \tilde{Q}_i^A\right|^2\right) + V(u,d,s,a_3,a_8)\Bigg)
\end{aligned} \tag{33}
$$

[5] We will also used ”normalized” roots $\mathbf{e}_0 = \boldsymbol{\alpha}_1/\sqrt{2}$ $\mathbf{e}_2 = \boldsymbol{\alpha}_2/\sqrt{2}$ and $\mathbf{e}_1 = \boldsymbol{\alpha}_{12}/\sqrt{2}$ and ”normalized” weights, e.g. $\mathbf{u} = \boldsymbol{\mu}_1/\sqrt{2}$.

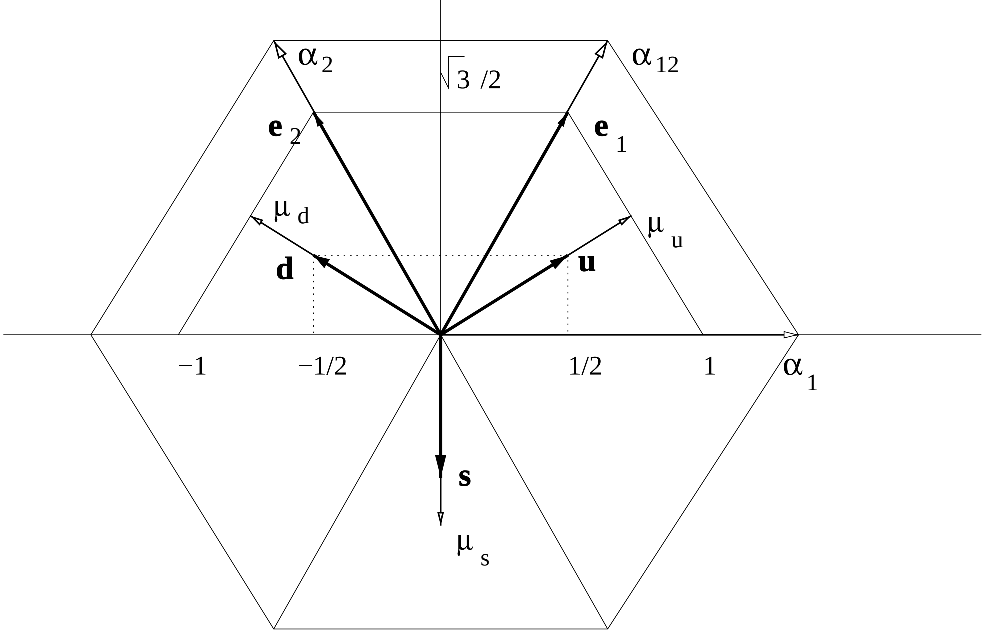

Figure 1: *Root and weight vectors in the Cartan plane for $SU(3)$ group. We have depicted explicitly the root vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ (the simple roots), the highest root $\boldsymbol{\alpha}_{12} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ and the weights of the fundamental representation $\boldsymbol{\mu}_1 \equiv \boldsymbol{\mu}_u$, $\boldsymbol{\mu}_2 \equiv \boldsymbol{\mu}_d$ and $\boldsymbol{\mu}_3 \equiv \boldsymbol{\mu}_s$, corresponding to u, d and s quarks respectively. We have also depicted the "normalized" roots (of unit length) $\mathbf{e}_1 = \boldsymbol{\alpha}_{12}/\sqrt{2}$ and $\mathbf{e}_2 = \boldsymbol{\alpha}_2/\sqrt{2}$ as well as normalized weights $\mathbf{u}$, $\mathbf{d}$ and $\mathbf{s}$ of the length $1/\sqrt{3}$.*

Here u, d, s label different *colors* of scalar components of quark supermultiplets Q_i while

$$\begin{aligned} \nabla^{(u,d)}_\mu &= \partial_\mu - \frac{i}{\sqrt{3}}\, A^{(u,d)}_\mu \\ \nabla^{(s)}_\mu &= \partial_\mu + \frac{i}{\sqrt{3}}\, A^{(u)}_\mu + \frac{i}{\sqrt{3}}\, A^{(d)}_\mu \end{aligned} \tag{34}$$

where we introduced also (non-orthogonal) components of the gauge fields interacting directly with u- and d-quarks

$$\begin{aligned} A^{(u)}_\mu &= \frac{\sqrt{3}}{2} A^{(3)}_\mu + \frac{1}{2} A^{(8)}_\mu \\ A^{(d)}_\mu &= -\frac{\sqrt{3}}{2} A^{(3)}_\mu + \frac{1}{2} A^{(8)}_\mu \end{aligned} \tag{35}$$

The potential in the Lagrangian (33)

$$\begin{aligned} & V(u,d,s,\mathbf{a}) = \\ &= g^2 \sum_{\boldsymbol{\alpha}\in\Delta_+} \left(\frac{1}{4} |D\boldsymbol{\alpha}|^2 + \left| \frac{\partial \mathcal{W}}{\partial \Phi_{\boldsymbol{\alpha}}} \right|^2 \right) + g^2 \mathbf{D}^2 + g^2 \left| \frac{\partial \mathcal{W}}{\partial \mathbf{a}} \right|^2 + \sum_{A=1}^{N_f} \sum_{i=u,d,s} \left(\left| \frac{\partial \mathcal{W}}{\partial Q^A_i} \right|^2 + \left| \frac{\partial \mathcal{W}}{\partial \tilde{Q}^A_i} \right|^2 \right) = \\ &= \frac{g^2}{4} \sum_{i\neq j} |D_{ij}|^2 + \frac{g^2}{12} \sum_{i<k} (D_{ii} - D_{kk})^2 + \frac{g^2}{2} \sum_{i\neq j} \frac{\partial \mathcal{W}}{\partial \Phi_{ij}} \overline{\left(\frac{\partial \mathcal{W}}{\partial \Phi_{ji}} \right)} + \\ &+ g^2 \left(\left| \frac{\partial \mathcal{W}}{\partial a_3} \right|^2 + \left| \frac{\partial \mathcal{W}}{\partial a_8} \right|^2 \right) + \sum_{A=1}^{N_f} \sum_{i=u,d,s} \left(\left| \frac{\partial \mathcal{W}}{\partial Q^A_i} \right|^2 + \left| \frac{\partial \mathcal{W}}{\partial \tilde{Q}^A_i} \right|^2 \right) \end{aligned} \tag{36}$$

$Q_1 = u$, $Q_2 = d$, $Q_3 = s$ and $a_{3,8}$ (31)

$$
\begin{aligned}
V(u,d,s,a_3,a_8) = {} & \frac{g^2}{4}\sum_{i\neq j} D_{ij}D_{ji} + g^2 \sum_{i\neq j} \bar{F}_{ij}F_{ji} + \\
& + \frac{g^2}{8}\left(D_{uu} - D_{dd}\right)^2 + \frac{g^2}{24}\left(D_{uu} + D_{dd} - 2D_{ss}\right)^2 + \\
& + \frac{g^2}{2}\left|F_{uu} - F_{dd} + \sqrt{2}\mu a_3\right|^2 + \frac{g^2}{2}\left|\frac{1}{\sqrt{3}}\left(F_{uu} + F_{dd} - 2F_{ss}\right) + \sqrt{2}\mu a_8\right|^2 + \\
& \sum_{A=1}^{N_f}\left(\frac{1}{2}\left|a_3 + \frac{a_8}{\sqrt{3}} + \sqrt{2}m_A\right|^2 \left(|u^A|^2 + |\tilde{u}^A|^2\right) + \frac{1}{2}\left|-a_3 + \frac{a_8}{\sqrt{3}} + \sqrt{2}m_A\right|^2 \left(|d^A|^2 + |\tilde{d}^A|^2\right) + \right. \\
& \left. + \frac{1}{2}\left|\frac{2}{\sqrt{3}}a_8 + \sqrt{2}m_A\right|^2 \left(|s^A|^2 + |\tilde{s}^A|^2\right)\right)
\end{aligned}
\tag{37}
$$

where color indices i, j as in (33) run over the set of u, d and s.

The QED coupling constant g in (33) is small at small μ, determined by the mass of light quarks and photons (cf. eq. (12))

$$
\frac{1}{g^2} \sim \log\frac{\mu m}{\Lambda^2}
\tag{38}
$$

With logarithmic accuracy we do not distinguish between two different coupling constants associated with two $U(1)$ factors in (33). Below we consider different types of $\mathcal{N} = 1$ vacua and analyse what kind of flux tubes do we have in each type.

3.2 r=1 vacua

Consider first $r = 1$ vacuum when only one $Q_i^A = Q_i^1$ quark flavor develops VEV. We drop flavor index below in this subsection because all other $r = 1$ vacua have the similar structure. As we have learnt in sect. 2.2 the VEV's of scalar fields in this vacuum are given by (13) where we assume for simplicity that μ and m are real and positive and choose the phase of the u-quark condensate to vanish. The adjoint scalar develop VEV given by (11), in terms of fields a_3 and a_8 this reads

$$
\begin{aligned}
\langle a_3 \rangle &= -\frac{3}{2\sqrt{2}}m \\
\langle a_8 \rangle &= -\frac{\sqrt{3}}{2\sqrt{2}}m
\end{aligned}
\tag{39}
$$

From the kinetic term for u-quark in (33) one can get the mass matrix for the $U(1)$ gauge fields $(A_\mu^{(3)}, A_\mu^{(8)})$

$$
\mathcal{M}_\gamma^2 = \frac{3}{2}g^2\mu\, m \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3} \end{pmatrix}
\tag{40}
$$

whose eigenvalues are

$$
m_1^2 = 2g^2\mu\ m = \frac{2}{3}g^2\zeta
\tag{41}
$$

together with

$$
m_2^2 = 0
\tag{42}
$$

where we have used parameter $\zeta = 2\langle\tilde{u}u\rangle = 3\mu m$. These equations show that one massive photon appears since u-quark develops nonzero VEV (13) and breaks one of two $U(1)$ gauge groups while the massless photon is associated with the second unbroken $U(1)$ group. This fact we have already pointed out in sect. 2.2. Classically the $SU(2)$ group which includes the latter $U(1)$ factor remains unbroken, see for example formula (11). However, in quantum theory this $SU(2)$ subgroup is broken down due to the Seiberg-Witten mechanism [4, 5] and the second photon acquires small mass of the order of $\sqrt{\mu\Lambda_{SU(2)}}$ due to monopole/dyon condensation in this $SU(2)$ sector.

on the two last lines in potential (37). In this approximation it coincides with the photon mass matrix (40) and therefore one complex a-field remains massless while the other one acquires the same mass (41) as one of the photons. This is directly related to the fact that $\mathcal{N}=2$ supersymmetry is *not* broken in the effective low energy QED (33) in the leading order in μ/m [7, 14]. To see this note, that in this approximation the perturbation of superpotential (1) proportional to μ is linear in fluctuations of a-fields. Thus it boils down to the Fayet-Iliopoulos (FI) F-term which does not break $\mathcal{N}=2$ supersymmetry in the effective QED [7, 14]. In the next to leading order in μ/m (which corresponds to taking into account fluctuations of the a-fields in the third line of potential (37)) $\mathcal{N}=2$ supersymmetry is broken and $\mathcal{N}=2$ supermultiplets split [14]. Below we restrict ourselves to the leading order in μ/m so that $\mathcal{N}=2$ supersymmetry is preserved in the effective low energy Abelian theory (33).

To complete the study of the mass spectrum in vicinity of $r=1$ vacuum consider the mass matrix for quarks. The d- and s-quarks are very heavy (with masses of the order of $m \gg \Lambda$) and, hence, decouple from the low-energy theory (see last two terms in (37)). The mass matrix for remaining four real components of u-quark $(\mathrm{Re}u, \mathrm{Re}\tilde{u}, \mathrm{Im}u, \mathrm{Im}\tilde{u})$ can be read off the terms

$$\frac{g^2}{6}D_{uu}^2 + \frac{g^2}{2}\left|F_{uu}+\sqrt{2}\mu a_3\right|^2 + \frac{g^2}{2}\left|\frac{F_{uu}}{\sqrt{3}}+\sqrt{2}\mu a_8\right|^2 = \frac{2g^2}{3}\left(\left|F_{uu}-\frac{3}{2}\mu m\right|^2+\frac{1}{4}D_{uu}^2\right)$$

of the potential (37) and has the form

$$\mathcal{M}_u^2 = 2g^2\mu\, m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{43}$$

with one zero eigenvalue corresponding to the state "eaten" by Higgs mechanism, and three other eigenvalues coinciding with the photon mass (41). These three real states of u-quark combine with two real states of massive a-field and three states of massive photon to form the bosonic part of one *long* $\mathcal{N}=2$ supermultiplet which contains eight boson and eight fermion states. The long multiplet appears because electric charges are screened in the broken $U(1)$ sector by Higgs condensate, therefore the corresponding central charges of $\mathcal{N}=2$ algebra vanish, and short BPS multiplets cannot appear [14]. To sum up, one gets one long $\mathcal{N}=2$ multiplet with mass (41) formed by the u-quark condensation and another one with much smaller mass of the order of $\sim\sqrt{\mu\Lambda_{SU(2)}}$ formed by the monopole/dyon condensation in the classically unbroken $SU(2)$ sector.

3.3 r=2 vacua in the low energy effective theory

Let us consider now the $r=2$ vacua of sect. 2.3. Assume for simplicity that we have only two quark flavors with masses m_1 and m_2. In this vacuum quark fields develop VEV's (19) while adjoint VEV's are given by (18). In a_3, a_8 basis they are

$$\begin{aligned} \langle a_3\rangle &= -\frac{1}{\sqrt{2}}(m_1-m_2) \\ \langle a_8\rangle &= -\sqrt{\frac{3}{2}}(m_1+m_2) \end{aligned} \tag{44}$$

The mass matrix for the gauge fields $(A_\mu^{(3)}, A_\mu^{(8)})$ can be read off the kinetic terms for u- and d-quarks in (33) and has the form

$$\mathcal{M}_\gamma^2 = \frac{g^2\xi}{2}\begin{pmatrix} 1+\omega & \frac{1}{\sqrt{3}}(1-\omega) \\ \frac{1}{\sqrt{3}}(1-\omega) & \frac{1}{3}(1+\omega) \end{pmatrix} \tag{45}$$

where we have introduced the parameters of $r=2$ vacua

$$\xi \equiv 2\langle \tilde{u}_1 u^1\rangle = 2\mu(2m_1+m_2) \tag{46}$$

$$\omega \equiv \frac{\langle \tilde{d}_2 d^2 \rangle}{\langle \tilde{u}_1 u^1 \rangle} = \frac{2m_2 + m_1}{2m_1 + m_2} \tag{47}$$

analogous to the parameter ζ of $r = 1$ vacuum (13). Two eigenvalues of this mass matrix are given by

$$\left(m_\gamma^2\right)_\pm = \frac{g^2 \xi}{2} \Omega_\pm \tag{48}$$

with

$$\Omega_\pm = \frac{2}{3}\left(1 + \omega \pm \sqrt{1 - \omega + \omega^2}\right) \tag{49}$$

We see that for generic values of m_1 and m_2 both $U(1)$ groups are broken and both photons acquire masses. The mass matrix for two complex fields a_3 and a_8 is identical to (45) in the leading order in μ/m as can be seen again from (37).

The mass matrix for quarks is now of the size 8×8 including four (real) components of u^1-quark and four components of d^2-quark. It has two zero eigenvalues associated with the two states "eaten" by the Higgs mechanism for two $U(1)$ gauge factors and two non-zero eigenvalues coinciding with photon masses (48). Each of these non-zero eigenvalues corresponds to three quark eigenvectors. Altogether we have two long $\mathcal{N} = 2$ multiplets with masses (48), each one containing eight bosonic and eight fermionic states.

Now let us briefly comment on the special case of coinciding masses $m_1 = m_2$ ($\omega = 1$) to be considered in detail in sect. 6. In this case $\langle a_3 \rangle = 0$ and $SU(2)$ subgroup of the $SU(3)$ gauge group is restored on the Coulomb branch at zero μ at least classically, see (18). However when one switches on the terms proportional to μ this $SU(2)$ subgroup becomes broken completely by u- and d-quark condensates. It is easy to see that all three masses of $SU(2)$ gauge fields are the same and given by (48) with $\Omega_+ = 2$ which corresponds to the special value $\omega = 1$ in (49) when mass matrix becomes diagonal.

4 Flux tubes

In this section we consider flux tubes in isolated $\mathcal{N} = 1$ charge vacua of the theory at $m_A \gg \Lambda$ and $\mu \ll \Lambda$. In this regime the low energy effective theory reduces to Abelian model (33) at weak coupling so one can use the semi-classical methods to study string solutions.

4.1 ANO strings in r=1 vacua

At $r = 1$ vacuum d- and s-quarks are heavy and we can simply ignore them. The VEV's of light fields are given by eqs. (13) and (39), in particular u- and $\tilde{u}$-quarks have the same VEV's. This suggest that one can look for the ANO string solution using the ansatz (13), (39), i.e. to fix

$$\begin{gathered} d = s = \tilde{d} = \tilde{s} = 0 \\ a_3 = -\frac{3}{2\sqrt{2}} m, \; a_8 = -\frac{\sqrt{3}}{2\sqrt{2}} m \end{gathered} \tag{50}$$

and express u and $\tilde{u}$ in terms of a single complex field φ

$$u = \bar{\tilde{u}} = \frac{\varphi}{\sqrt{2}} \tag{51}$$

Since u-quark interacts only with particular combination $A_\mu^{(u)}$ of the gauge fields $A_\mu^{(3)}$ and $A_\mu^{(8)}$ given by (35) it is natural to assume that only this combination is non-zero on corresponding string solution. To implement this let us rotate the fields $A_\mu^{(3)}$ and $A_\mu^{(8)}$ to another orthogonal basis

$$\begin{aligned} A_\mu^{(u)} &= \frac{\sqrt{3}}{2} A_\mu^{(3)} + \frac{1}{2} A_\mu^{(8)} \\ A_{2\mu} &= -\frac{1}{2} A_\mu^{(3)} + \frac{\sqrt{3}}{2} A_\mu^{(8)} \end{aligned} \tag{52}$$

The meaning of the subscript "2" of $A_{2\mu}$ means that it is directed along $\boldsymbol{\alpha}_2 \sim \mathbf{e}_2$ (see fig. 1) and at the moment we only need that

$$A_{2\mu} = 0 \tag{53}$$

ansatz the bosonic part of softly broken $\mathcal{N} = 2$ QED (33) reduces to the form of standard Abelian Higgs model (the relativistic version of the Landau-Ginzburg model)

$$S_{\rm AH} = \int {\rm d}^4x \left(\frac{1}{4g^2} \left(F^{(u)}_{\mu\nu} \right)^2 + |\nabla^{(u)}_\mu \varphi|^2 + \lambda \left(|\varphi|^2 - \zeta \right)^2 \right) \tag{54}$$

with particular value of quartic coupling and electric charge $n_e = 1/\sqrt{3}$ of the field φ, see (34). The field φ develops VEV (13) $|\langle\varphi\rangle|^2 = \zeta = 3|\mu m|$, therefore U(1) gauge group is broken, or the photon acquires mass $m_\gamma^2 = 2g^2\zeta/3$ while the Higgs mass is equal to $m_H^2 = 4\lambda\zeta$. Note, that the photon mass coincides with that of (41), since the fields $A^{(u)}_\mu$ and $A_{2\mu}$ diagonalize the photon mass matrix (40), if it were not so the fields $A^{(u)}_\mu$ and $A_{2\mu}$ would mix leading to contradiction between condition (53) and equations of motion.

Strictly speaking the substitution $\langle a_3\rangle = -\frac{3}{2\sqrt{2}}m$ and $\langle a_8\rangle = -\frac{\sqrt{3}}{2\sqrt{2}}m$ does not satisfy equations of motion following from (33). In fact VEV's a_3 and a_8 (50) get x-dependent corrections of the order of $\sqrt{\mu/m}$ [7, 14]. However, as we already explained in sect. 3.2 one can neglect this effect in the leading order in μ/m and consider ζ to be just a constant $\zeta = 3\mu m$. In this approximation the perturbation term in the superpotential (1) is linear in a and reduces to the FI F-term which does not break $\mathcal{N} = 2$ supersymmetry in effective QED [7, 14]. As is explained in detail in [14] the ANO strings in $U(1)$ theory are BPS-saturated in this limit (see also [19]). Below we use the term "BPS string" for these "almost BPS" solutions, they belong to the short $\mathcal{N} = 2$ multiplets which become long $\mathcal{N} = 1$ multiplets when next to leading order corrections (breaking $\mathcal{N} = 2$ supersymmetry) are taken into account [14].

Let us now briefly remind the basic features of the BPS ANO strings [11]. For arbitrary λ in (54) the Higgs mass m_H (the inverse correlation length) and photon mass m_γ (the inverse penetration depth) are different and their ratio is important parameter in theory of superconductivity, characterizing the type of superconductor. Namely, for $m_H < m_\gamma$ one has type I superconductor, while for $m_H > m_\gamma$ it is of type II, this is related to the fact that scalar field produces an attraction for two vortices, while the electromagnetic field produces a repulsion. The boundary separating superconductors of the I and II type corresponds to $m_H = m_\gamma$, i.e. to special value of quartic coupling λ

$$\lambda \ = \frac{n_e^2 g^2}{2} = \frac{g^2}{6} \tag{55}$$

This is exactly the value of quartic coupling in (54) we get from the potential (37) using the ansatz (50), (51). In this case vortices do not interact. It is well known that vanishing of interaction when $m_H = m_\gamma$ can be explained by the BPS nature of the ANO strings. The ANO string satisfies the first order equations and saturate the Bogomolny bound [11], which can be found from the following representation of string tension T (see (54)),

$$T = 2\pi\zeta\,|n| + \int d^2x \left\{ \left(\frac{1}{2g} F_{IJ} \pm \frac{g}{2\sqrt{3}} \left(|\varphi|^2 - \zeta \right) \epsilon_{IJ} \right)^2 + \frac{1}{2} \left| \nabla_I\, \varphi \pm i\epsilon_{IJ} \nabla_J\, \varphi \right|^2 \right\} \tag{56}$$

Here indices $I, J = 1, 2$ denote coordinates transverse to the axis of the vortex. For positive n we take the upper sign in (56), whereas for negative n we take the lower sign. The minimal value of the tension is reached when both positive terms in the integrand of (56) vanish, then the string tension becomes

$$T_{BPS} \ = \ 2\pi\zeta\,|n| \tag{57}$$

where the winding number n counts the magnetic flux $2\pi n$. The linear dependence of string tensions on n is consistent with the absence of string interactions.

For the $n = 1$ case vanishing of the integrand in (56) leads to well-known two first order differential equations

$$\begin{aligned} r\frac{\rm d}{{\rm d}r}\,\phi(r) - f(r)\,\phi(r) \ &= \ 0 \\ -\frac{1}{r}\,\frac{\rm d}{{\rm d}r} f(r) + \frac{g^2}{3}\,\left(\phi^2(r) - \zeta \right) \ &= \ 0 \end{aligned} \tag{58}$$

(for the positive signs in (56)), where the profile functions $\phi(r)$ and $f(r)$ are introduced in a standard way, i.e.

$$\begin{aligned} \varphi(x) &= \phi(r)\,{\rm e}^{i\,\vartheta} \\ A^{(u)}_I(x) &= \sqrt{3}\epsilon_{IJ}\,\frac{x_J}{r^2}\,\left[f(r) - 1 \right] \end{aligned} \tag{59}$$

transverse to the axis of vortex $(1,2)$-plane. The profile functions are real and satisfy the boundary conditions

$$\begin{aligned} \phi(0) &= 0\ , & f(0) &= 1 \\ \phi(\infty) &= \sqrt{\zeta}\ , & f(\infty) &= 0 \end{aligned} \tag{60}$$

which ensures that scalar field reaches its VEV $\sqrt{\zeta}$ at infinity and vortex carries one unit of magnetic flux. Equations (58) with boundary conditions (60) lead to unique solution for the profile functions (although an analytic form of this solution is not found, though for $\zeta = 0$ the system (58) is equivalent to the "radial" Liouville equation). The tension of string with winding number $n = 1$ is given by particular case of (57)

$$T_{r=1} = 2\pi\zeta = 6\pi|\mu m| \tag{61}$$

In $\mathcal{N} = 2$ QED emergence of the first order equations (58) means that some (half) of the SUSY charges of $\mathcal{N} = 2$ algebra act trivially onto the ANO solution (cf. [16, 17, 18, 14]). In this case the Bogomolny (topological) bound for the string tension coincides with the central charge of SUSY algebra.

Now let us discuss the embedding of flux (winding) numbers of the string (50), (51), (59) into the Cartan subalgebra of $SU(3)$ group. Throughout this paper we will use the convention of labeling the flux of a given ANO string by magnetic charge of the monopole which produces this flux and can be attached to its end [6], this is possible since both string fluxes and monopole charges are elements of the group $\pi_1(U(1)^{\otimes 2}) = \mathbf{Z}^{\otimes 2}$. This convention is convenient because specifying the flux of given string we automatically fix the charge of monopole confined by this string.

The string solution (50), (51), (59) has non-zero $A_\mu^{(u)}$ while $A_{2\mu}$ vanishes, i.e. the matrix A_μ for this configuration looks like

$$A_\mu = \frac{A_\mu^{(u)}}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{62}$$

It preserves the $SU(2)$ subgroup acting in the 2×2 right low corner of this matrix. Thus the flux of this string (charge of the monopole attached to its end) is orthogonal to the root vector $\boldsymbol{\alpha}_2$ (151) (or $\mathbf{e}_2$, see fig. 1). Hence, the string charge vector is proportional to u-quark weight vector on the Cartan plane, $\boldsymbol{\mu}_u \sim \boldsymbol{\mu}_1$. We call this string u-string and for this reason denote its charge vector $\mathbf{q}_u$.

To work out the absolute value of the u-string charge we recall that the Dirac quantization condition in the $r = 1$ vacuum with non-zero VEV of the u-quark looks like

$$\mathbf{q}_u \mathbf{u} = \frac{n}{2} \tag{63}$$

Thus for winding number $n = 1$ one has

$$\mathbf{q}_u = \frac{3}{2}\mathbf{u} \tag{64}$$

where we use that $|\mathbf{u}| = n_e = 1/\sqrt{3}$, see (34).

From (64) we see that absolute value of the string charge is $|\mathbf{q}_u| = \sqrt{3}/2$ and this vector points into the middle of the side of adjoint hexagon, see fig. 2. Note, that monopole charges (whose positions in the Cartan plane coincide with the positions of W-boson charges) correspond to the corners of adjoint hexagon. We see that the component of monopole charge parallel to vector $\mathbf{u}$ is confined by the ANO $\mathbf{u}$-string. Moreover, it is easy to see from fig. 2 that the values of projections of charges of $\mathbf{e}_1$ and $\mathbf{e}_0$ monopoles onto the $\mathbf{u}$-direction exactly matches the charge of the $\mathbf{u}$-string. Thus $\mathbf{u}$-string confines $\mathbf{u}$-component of $\mathbf{e}_1$ and $\mathbf{e}_0$ monopole charges. This remark can be used as an explanation of rather strange from group theory point of view value of charge of the u-string lying outside the root lattice of $SU(3)$ group.

However, all monopoles present in the theory ($\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_0$- monopoles) have also non-zero component in the orthogonal direction to vector $\mathbf{u}$, i.e. along $\mathbf{e}_2$. What happens to this component of the monopole charge? The answer to this question can be found in quantum theory, where the story is a bit more complicated. As we already explained in sect. 2.2 the $SU(2)$ gauge subgroup is preserved only classically at $r = 1$ vacua, and from the Seiberg-Witten exact solution [4, 5] we know that in quantum theory $SU(2)$-subsector runs into

[6]This monopole can be a superposition of "really existing" monopoles in given vacuum as we will see below.

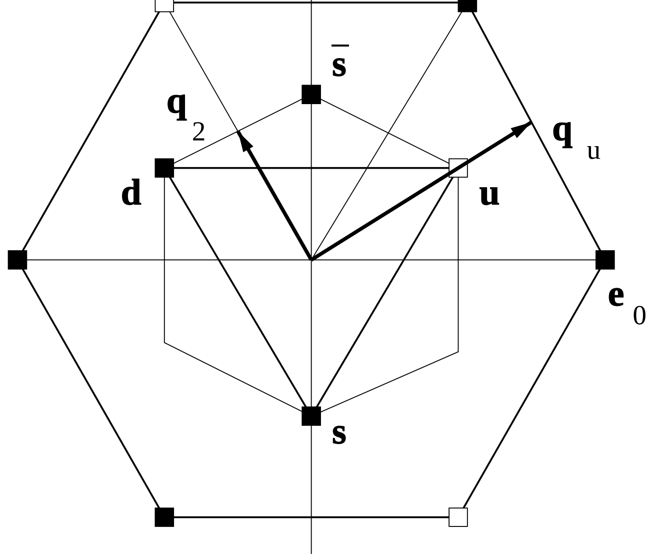

Figure 2: *Strings and different phases in $r = 1$ vacuum. Black squares correspond to the monopole and quark states in confinement phase while white squares – to the states in Higgs phase. Vectors $\mathbf{q}_u$ and $\mathbf{q}_2 = \mathbf{e}_2/2$ label the (charges of) magnetic and electric strings correspondingly.*

strong coupling regime where the $SU(2)$ subgroup is never restored; instead, one gets either monopole or dyon vacuum. Consider, say, the monopole one. At nonzero μ monopole develops VEV of the order of $\mu\Lambda_{SU(2)}$ and this monopole has charge $\mathbf{e}_2$, see (14). It means that $\mathbf{e}_2$-components of monopole charges are screened by the condensation of the $\mathbf{e}_2$-monopole.

Passing to dual magnetic theory at strong coupling one can also study the formation of the ANO *electric* string due to condensation of $\mathbf{e}_2$-monopole along same lines as above. It is clear from the Dirac quantization condition that $\mathbf{e}_2$-string arises with the charge $\mathbf{q}_2 = \mathbf{e}_2/2$, see fig. 2. It confines $\mathbf{e}_2$-components of d- and s-quark electric charges while their $\mathbf{u}$-component is screened by the u-quark condensation. Note again, that $\mathbf{e}_2$-components of d- and $\bar{s}$-quark electric charges exactly coincides with the charge of the $\mathbf{e}_2$-string. The electric $\mathbf{e}_2$-string which confines the fundamental charge is very similar to strings at monopole vacua of $SU(3)$ gauge theory without fundamental matter (labeled by $r = 0$ in our notations) studied in [15].

To sum up, in each $r = 1$ vacuum we obtain one BPS magnetic ANO $\mathbf{u}$-string and one BPS electric $\mathbf{e}_2$-string. In other words we have mixed Higgs phase for u-quark and $\mathbf{e}_2$-monopole and confining phase for $\mathbf{e}_1$- and $\mathbf{e}_0$-monopole as well as for d- and s-quark, see fig. 2.

4.2 Strings in r=2 vacua

Now let us turn to strings in $r = 2$ vacua. For simplicity we assume, first, the presence of only two quark flavors with masses m_1 and m_2 under the same conditions as in sect. 3.3, i.e. we consider s-quark to be heavy and ignore it in low-energy theory. The VEV's of all fields are given by (19) and (44), in particular VEV's of u- and $\tilde{u}$-quarks are equal to each other as well as VEV's of d- and $\tilde{d}$-quarks. Therefore, we look for string solutions using the following ansatz

$$
\begin{aligned}
u^A &= \bar{\tilde{u}}_A = \delta_{A,1}\frac{\varphi_u}{\sqrt{2}} \\
d^A &= \bar{\tilde{d}}_A = \delta_{A,2}\frac{\varphi_d}{\sqrt{2}}
\end{aligned}
\tag{65}
$$

while the fields a_3 and a_8 in the leading order in μ/m are given by their VEV's.

With this ansatz the effective action (33) becomes

$$
\begin{aligned}
S = \int \mathrm{d}^4x \Bigg(&\frac{1}{4g^2}\left(F^{(3)}_{\mu\nu}\right)^2 + \frac{1}{4g^2}\left(F^{(8)}_{\mu\nu}\right)^2 + |\nabla^{(u)}_\mu \varphi_u|^2 + |\nabla^{(d)}_\mu \varphi_d|^2 + \\
&\frac{g^2}{8}\left(|\varphi_u|^2 - |\varphi_d|^2 - \xi(1-\omega)\right)^2 + \frac{g^2}{24}\left(|\varphi_u|^2 + |\varphi_d|^2 - \xi(1+\omega)\right)^2\Bigg)
\end{aligned}
\tag{66}
$$

and (47) (note that we consider both of them real and positive). Gauge fields $A_\mu^{(u)}$ and $A_\mu^{(d)}$ here as functions of orthogonal fields $A_\mu^{(3)}$ and $A_\mu^{(8)}$ are given by (35). To study more general case of complex ξ and ω one should modify ansatz (65) taking into account relative phases of the fields Q and $\tilde{Q}$.

Let us now derive the Bogomolny bound for string solutions in the theory (66). Assuming that all fields depend only on two spatial coordinates orthogonal to the string axis one can rewrite (66) as follows

$$
\begin{aligned}
T = \int d^2x \Bigg(&\left[\frac{1}{2g} F_{IJ}^{(3)} \pm \frac{g}{4} \left(|\varphi_u|^2 - |\varphi_d|^2 - \xi(1-\omega) \right) \epsilon_{IJ} \right]^2 + \\
&+ \left[\frac{1}{2g} F_{IJ}^{(8)} \pm \frac{g}{4\sqrt{3}} \left(|\varphi_u|^2 + |\varphi_d|^2 - \xi(1+\omega) \right) \epsilon_{IJ} \right]^2 + \\
+\frac{1}{2} \left| \nabla_I^{(u)} \varphi_u \pm i\epsilon_{IJ} \nabla_J^{(u)} \varphi_u \right|^2 &+ \frac{1}{2} \left| \nabla_I^{(d)} \varphi_d \pm i\epsilon_{IJ} \nabla_J^{(d)} \varphi_d \right|^2 \pm \frac{1}{\sqrt{3}} \left(\tilde{F}^{(u)} + \tilde{F}^{(d)} \omega \right) \xi \Bigg)
\end{aligned}
\tag{67}
$$

where we introduced (two-dimensional) dual field strength for gauge fields $A_\mu^{(u,d)}$ as $\tilde{F}^{(u,d)} = \frac{1}{2}\epsilon_{IJ} F_{IJ}^{(u,d)}$. The upper sign here corresponds to positive total flux, the value of $\frac{1}{\sqrt{3}} \int d^2x \left(\tilde{F}^{(u)} + \tilde{F}^{(d)} \omega \right)$, while the lower sign corresponds to the negative total flux.

The last term in (67) represents exactly the string fluxes while all other positive terms in (67) should be zero on the BPS solutions, leading to the following first order equations

$$
\begin{aligned}
\frac{1}{2g} F_{IJ}^{(3)} + \frac{g}{4} \varepsilon \left(|\varphi_u|^2 - |\varphi_d|^2 - \xi(1-\omega) \right) \epsilon_{IJ} = 0 \\
\frac{1}{2g} F_{IJ}^{(8)} + \frac{g}{4\sqrt{3}} \varepsilon \left(|\varphi_u|^2 + |\varphi_d|^2 - \xi(1+\omega) \right) \epsilon_{IJ} = 0 \\
\nabla_I^{(u)} \varphi_u + i\varepsilon \epsilon_{IJ} \nabla_J^{(u)} \varphi_u = 0 \\
\nabla_I^{(d)} \varphi_d + i\varepsilon \epsilon_{IJ} \nabla_J^{(d)} \varphi_d = 0
\end{aligned}
\tag{68}
$$

where $\varepsilon = \pm$ is the sign of total flux.

One can classify possible solutions to these equations by behavior of the fields at spatial infinity in $(1, 2)$ plane. One type of solutions has nontrivial winding for the u-quark another type has winding for the d-quark and there are also mixed solutions when both u- and d-quarks wind at infinity.

Let us start with string solutions when only u-quark winds. Assuming for simplicity the unit winding number one has at $x \to \infty$

$$
\begin{aligned}
\varphi_u &\sim e^{i\vartheta} \sqrt{\xi} \\
\varphi_d &\sim \sqrt{\omega \xi}
\end{aligned}
\tag{69}
$$

This behavior requires the following behavior at infinity for the gauge fields

$$
\frac{1}{\sqrt{3}} A_I^{(u)} \underset{r\to\infty}{\longrightarrow} \partial_I \vartheta \tag{70}
$$

and

$$
A_I^{(d)} \underset{r\to\infty}{\longrightarrow} 0 \tag{71}
$$

which ensures finite contribution to string tension coming from kinetic terms of u- and d-quarks. Such behavior at infinity means that the flux of the field $A_\mu^{(d)}$ vanishes while the flux of the $A_\mu^{(u)}$ field is

$$
\frac{1}{\sqrt{3}} \int d^2x \tilde{F}^{(u)} = 2\pi \tag{72}
$$

for unit winding number, then representation (67) for the BPS string tension gives

$$
T_1 = 2\pi\xi \;=\; 4\pi \; |\mu(2m_1 + m_2)| \tag{73}
$$

(given by (35)) and $A_{1\mu}$ (the component along the vector $\mathbf{e}_1$) as

$$A_{1\mu} = \frac{1}{2}A_\mu^{(3)} + \frac{\sqrt{3}}{2}A_\mu^{(8)}$$
$$A_\mu^{(d)} = -\frac{\sqrt{3}}{2}A_\mu^{(3)} + \frac{1}{2}A_\mu^{(8)} \tag{74}$$

and rewrite our first order eqs. (68) in terms of these fields.

The behavior at infinity (69)-(71) suggests that one can look for a solution in terms of the following profile functions

$$\varphi_u(x) = \phi_u(r)\mathrm{e}^{i\,\vartheta}, \;\; \varphi_d(x) = \phi_d(r)$$
$$A_{1I}(x) = 2\epsilon_{IJ}\,\frac{x_J}{r^2}\,\left[f_1(r)-1\right], \;\; A_I^{(d)}(x) = \sqrt{3}\epsilon_{IJ}\,\frac{x_J}{r^2}\,f_d(r) \tag{75}$$

Note, that the charge of u-quark with respect to $A_{1\mu}$ field is $n_e = 1/2$, while the charge of d-quark with respect to $A_\mu^{(d)}$ is $n_e = 1/\sqrt{3}$, which exactly corresponds to the coefficients in (75). With this substitution the first order equations (68) turn into the system of four first-order nonlinear differential equations

$$\begin{aligned}
r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_u(r) - f_1(r)\,\phi_u(r) \;+\frac{1}{2}f_d(r)\,\phi_u(r) &= \;0\\
r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi_d(r) - f_d(r)\,\phi_d(r) \; &= 0\\
-\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}f_1(r) + \frac{g^2}{4}\,\left(\phi_u^2(r)-\xi\right) \; &= \;0\\
-\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}f_d(r) + \frac{g^2}{3}\,\left(\phi_d(r)^2 - \frac{1}{2}\phi_u(r)^2 + \frac{1}{2}\xi - \omega\xi\right) \; &= \;0
\end{aligned} \tag{76}$$

The reason why the fields $\varphi_d - \sqrt{\omega\xi}$ and $A_\mu^{(d)}$ with trivial behavior at infinity cannot be simply put to zero is mixing between φ_u, $A_{1\mu}$ and φ_d, $A_\mu^{(d)}$ since the fields φ_u and $A_{1\mu}$ which wind at infinity are not eigenvectors of the mass matrix (45). In other words (76) does not have nontrivial solutions with $f_d \equiv 0$ and $\phi_d \equiv \sqrt{\omega\xi}$ (which is a naive ansatz suggested by the boundary conditions (69), (71)).

First order equations (76) should be supplemented by the boundary conditions which ensure that quark fields go to their VEV's at infinity, u-quark field has no singularity at $r=0$ ($\phi_u(0)=0$) and $A_{1\mu}$ field carries flux equal to 2π ($f_1(0)=1$).

Now let us discuss the charges of our string. Since $A_\mu^{(d)}$ goes to zero at infinity the gauge field matrix looks like

$$A_\mu \underset{r\to\infty}{\longrightarrow} \frac{A_{1\mu}}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{77}$$

at infinity. Eq. (77) means that the charge of string is directed along the root $\boldsymbol{\alpha}_{12} \sim \mathbf{e}_1$ in the Cartan algebra. Its charge vector should satisfy the Dirac quantization condition $\mathbf{q}_1\mathbf{u} = 1/2$ (u-quark winds) which gives

$$\mathbf{q}_1 = \mathbf{e}_1 \tag{78}$$

This is the reason why we call this string $\mathbf{e}_1$-string. Its charge is directed to the upper right corner of the adjoint hexagon, see fig. 3.

In a similar way one can consider the string solution when d-quark winds at infinity, while u-quark field runs to its VEV. This gives us $\mathbf{e}_2$-string with charge vector directed to the upper left corner of the adjoint hexagon, see fig. 3. The easiest way to describe this string is to use another set of orthogonal gauge potentials $A_{2\mu}$ and $A_\mu^{(u)}$ related to $A_{1\mu}$ and $A_\mu^{(d)}$ by

$$A_\mu^{(u)} = \frac{\sqrt{3}}{2}A_{1\mu} - \frac{1}{2}A_\mu^{(d)}$$
$$A_{2\mu} = \frac{1}{2}A_{1\mu} + \frac{\sqrt{3}}{2}A_\mu^{(d)} \tag{79}$$

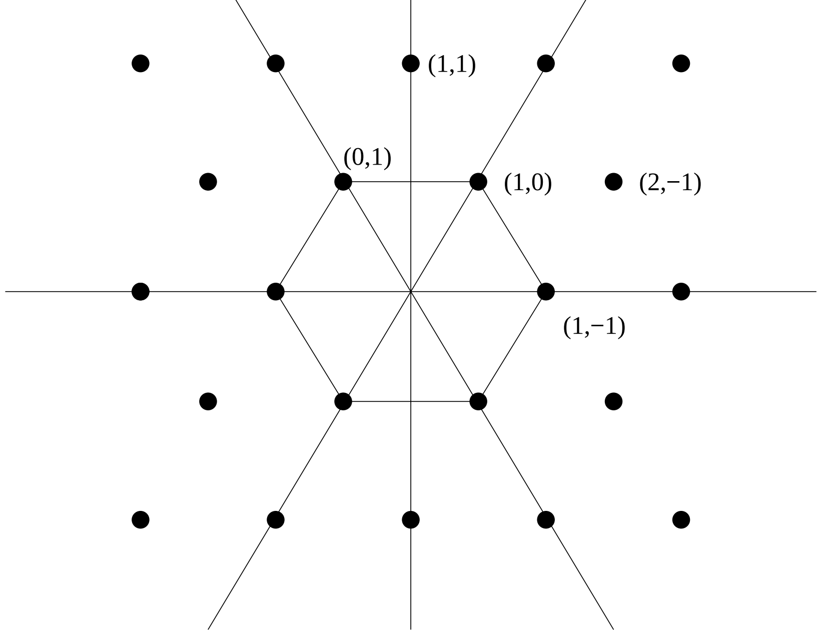

Figure 3: *Lattice of string solutions in* $r = 2$ *vacuum. We have specified explicitly* $(1,0) = \mathbf{e}_1$, $(0,1) = \mathbf{e}_2$, $(1,-1) = \mathbf{e}_0$, $(1,1) = \mathbf{e}_1 + \mathbf{e}_2$ *and* $(2,-1) \equiv 2\mathbf{u}$ *strings.*

The behavior at infinity of $\mathbf{e}_2$-string is given by

$$\begin{gathered}\varphi_d(x) = \phi_d(r)\,\mathrm{e}^{i\,\vartheta},\ \varphi_u(x) = \phi_u(r) \\ A_{2I}(x) = 2\epsilon_{IJ}\,\frac{x_J}{r^2}\,(f_2(r)-1)\,,\ A_I^{(u)}(x) = \sqrt{3}\epsilon_{IJ}\,\frac{x_J}{r^2}\ f_u(r)\end{gathered} \tag{80}$$

The profile functions here satisfy equations and boundary conditions similar to (76). The tension of the $\mathbf{e}_2$-string is given by

$$T_2 = \ 2\pi\omega\xi \ = \ 4\pi\ |\mu(2m_2 + m_1)| \tag{81}$$

and this string, if exists, is also BPS saturated.

In general, one can also consider string solutions when both u- and d-quarks wind. Let us fix some basis, say, the potentials $A_{1\mu}$ and $A_\mu^{(d)}$ (74) and introduce the profile functions for the string when u-quark winds n times and d-quark winds k times (to be denoted as (n,k)-string since its charge in our normalization is $\mathbf{q}_{n,k} = n\mathbf{e}_1 + k\mathbf{e}_2$) defined as

$$\begin{gathered}\varphi_u(x) = \phi_u(r)\,\mathrm{e}^{in\vartheta},\ \varphi_d(x) = \phi_d(r)\mathrm{e}^{ik\vartheta}, \\ A_{1I}(x) = 2\varepsilon\epsilon_{IJ}\,\frac{x_J}{r^2}\ \ (f_1(r) - f_1(0))\,,\ A_I^{(d)}(x) = \sqrt{3}\ \varepsilon\epsilon_{IJ}\,\frac{x_J}{r^2}\ \ (f_d(r) - f_d(0))\end{gathered} \tag{82}$$

These profile functions satisfy the first order equations (76), the only modification is due to the extra sign factor ε in (82) which accounts for possible positive or negative total flux

$$\varepsilon = \varepsilon_{n,k} = \frac{n + \omega k}{|n + \omega k|} = \mathrm{sign}(n + \omega k) = \pm 1 \tag{83}$$

The boundary conditions for profile functions of (n,k)-string read

$$\phi_u(0) = 0,\ \phi_d(0) = 0,\ \phi_u(\infty) = \sqrt{\xi},\ \phi_d(\infty) = \sqrt{\omega\xi} \tag{84}$$

for the scalar fields and

$$f_1(0) = \varepsilon_{n,k}(n + \frac{k}{2}),\ f_d(0) = \varepsilon_{n,k}\,k,\ f_1(\infty) = 0,\ f_d(\infty) = 0 \tag{85}$$

for the gauge fields. The boundary condition for $f_1(0)$ is easy to derive from $f_u(0) = \varepsilon_{n,k}\,n$ using relations (79).

$$T_{n,k} = 2\pi\xi|n+\omega k| = 4\pi\,|\mu\,(n(2m_1+m_2)+k(2m_2+m_1))| \tag{86}$$

The lattice of possible BPS (n,k)-strings is shown in fig. 3; $\mathbf{e}_1$-string is identical to $(1,0)$-string while $\mathbf{e}_2$-string is nothing but $(0,1)$-string. There is one more string whose charge belongs to the adjoint hexagon, namely the $(1,-1)$-string (to be called also $\mathbf{e}_0$-string), see fig. 3. As is clear from fig. 3, $\mathbf{e}_1$-string can be considered as a bound state of $\mathbf{e}_0$ and $\mathbf{e}_2$-string, which exactly corresponds to decomposition of the root $\boldsymbol{\alpha}_{12}$ into linear combination of the simple roots $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, see fig. 1. It is clear, thus, that string charges are labeled by (normalized to unity) root vectors of the $SU(3)$ root lattice. We will return to this question in sect. 5, where we study which of the BPS (n,k)-strings exist as solutions to the first order equations (76) and which among them are stable. Here we conclude by considering certain special cases in which some of (n,k)-strings becomes ANO strings and equations (76) reduce to ANO equations (58) with one gauge and one scalar potential.

Consider, first, the simplest example of this kind with both quarks having unit winding number, namely the $(1,1)$- or $(\mathbf{e}_1+\mathbf{e}_2)$-string. This string can be considered as bound state of $\mathbf{e}_1$- and $\mathbf{e}_2$-strings. If this bound state exists and saturates the Bogomolny bound then (86) suggests for its tension

$$T_{1,1} = 12\pi\,|\mu(m_1+m_2)| \tag{87}$$

Now let us give some evidence that this BPS string indeed exists. Consider the special value of parameter (47) $\omega=1$, then VEV's of u- and d-quarks are equal. The string solution can be easily constructed with the only component of gauge field $A_\mu^{(8)}$ being non-zero (putting $A_\mu^{(3)}=0$) and one complex scalar field introduced via

$$\varphi_u = \varphi_d = \frac{\varphi}{\sqrt{2}} \tag{88}$$

With this ansatz the theory (66) reduces to standard Abelian Higgs model of type (54) with equal values for photon and scalar masses $m_\gamma = m_H = g^2\xi/3$ and this model obviously possesses the BPS ANO string solution. This mass of the Abelian Higgs model coincides with the eigenvalue of the photon mass matrix, corresponding to $\Omega = 2/3$, see (48), (49). This is one of possible explanations why a single gauge potential and single scalar field appear in the string solution at $\omega=1$: gauge field $A_\mu^{(8)}$ and scalar (88) correspond to eigenvectors of gauge and scalar mass matrices.

One can reach the same conclusion directly from equations (76). Substituting the ansatz $\phi_u=\phi_d$ and $f_1 = 3/2\ f_2$ (this is consistent with boundary conditions (85) because $f_1(0)=3/2$ and $f_d(0)=1$ for $(1,1)$-string) we see that four equations (76) at $\omega=1$ reduce exactly to the system of two ANO equations (58), under the following modifications: the $r=1$ parameter ζ is replaced by the $r=2$ parameter ξ and the coefficient in front of the second term in the last equation of (58) replaced by $m_\gamma^2/2\xi = g^2/6$ for $\omega=1$.

Now let us move parameter ω away from $\omega=1$. It is clear that under continuous deformation string solution cannot disappear or become non-BPS state, since BPS string belongs to a short multiplet which cannot turn into long one without breaking of some amount of supersymmetry (the number of states cannot jump). Thus, at least at some region of values of parameter ω we expect existence of the BPS $(\mathbf{e}_1+\mathbf{e}_2)$-string, we specify this region in sect. 5.3.

Consider $(2,-1)$-string at $\omega\to 0$. Substituting $\phi_d=0$ and $f_d = -\frac{2}{3}f_1$ into equations (76) (note that this is consistent with the boundary conditions $f_1=3/2$, $f_d=-1$) and defining $\tilde f = -2f_d = \frac{4}{3}f_1$ one gets exactly the ANO equations (58) (with ζ substituted by ξ) for the functions ϕ_u and $\tilde f$ (the correct value for the photon mass for $\omega=0$ will be $m_\gamma^2 = 2g^2\xi/3$, see (48), (49)); hence we see that $(2,-1)$-string turns into the ANO string at $\omega=0$. It means that solution to (76) for the BPS $(2,-1)$-string exists at least in some region of ω around zero. The charge of the $(2,-1)$-string is shown on fig. 3, this string has *double* charge (flux) compare to the $\mathbf{u}$-string considered in previous section (see fig. 2), therefore we may also call this string as $2\mathbf{u}$-string. Now one can address the following question: what is relation between $(2,-1)$-string in $r=2$ vacuum and $\mathbf{u}$-string of $r=1$ vacuum and why in $r=2$ vacuum we have only $2\mathbf{u}$-strings with the double flux?

To understand this relation let us remind that in the limit $\omega\to 0$ $r=2$ vacuum moves along the Coulomb branch towards the $r=1$ vacuum and at $\omega=0$ these two vacua coalesce. In this limit $(2,-1)$-string becomes the ANO string which actually coincides with the $2\mathbf{u}$-string of $r=1$ vacuum with the double flux directed along vector $\mathbf{u}$ in the Cartan plane, see fig. 2. The double flux appears since in the $r=2$ vacuum we require always that d-quark field winds k times, where k is an integer ($k=-1$ for the $(2,-1)$-string). Therefore by continuity we have only this $(2,-1)$-string in $r=2$ vacuum even in the limit $\omega\to 0$ in which d-quark condensate vanishes.

Instead, in $r=1$ vacuum we have always $\langle\tilde d d\rangle = 0$ and the condition for the continuity of the d-quark wave function does not appear at all. Therefore one can have $\mathbf{u}$-string with the half-integer flux in $r=1$ vacuum in $r=1$ vacuum and its absence in $r=2$ vacuum can be considered as a way to distinguish between these two vacua in the limit $\omega \to 0$ (see similar discussion about the Argyres-Douglas points in [20]). Similar to $(2,-1)$-string (or $2\mathbf{u}$-string) at $\omega = 0$ one can consider also $(-1,2)$-string (or $2\mathbf{d}$-string) at $\omega \to \infty$ (together with $\xi \to 0$ and keeping $\omega\xi = $ const). It is easy to see that in this limit eqs. (76) reduce to eqs.(58) and $(-1,2)$-string becomes ANO string. This means that the BPS solution for $(-1,2)$-string exists at large ω.

In this section we presented few indirect arguments to determine the BPS nature (and stability) of particular (n,k)-strings. More detailed analysis of the nature of (n,k)-strings and their stability, based on study of string interactions, will be presented in sect. 5.3. One may also use more direct evidence based on numerical simulations (we have done this using the MAPLE program) of the BPS first order equations (76) [7].

To conclude this section note, that electric flux tubes quite similar to our magnetic (n,k)-strings were found in $\mathcal{N}=2$ $SU(3)$ theory without fundamental matter at strong coupling vacua [15].

4.3 BPS formula for the string tensions

In this section we rewrite the mass formulas for the BPS strings studied in previous sections in a form which is more familiar for the BPS objects. It has general structure

$$T_{BPS} = 2\pi|\mathbf{q}_s\mathbf{f}| \tag{89}$$

determined by central charges of $\mathcal{N}=2$ algebra [16, 17, 18, 14]. Namely, $\mathbf{q}_s$ is the charge vector of a given string in the Cartan plane (see fig. 2 and fig. 3) and $\mathbf{f}$ is the generalized vector parameter of the FI F-term, which can be defined as

$$\mathbf{f} = -4\mu\boldsymbol{\phi} \tag{90}$$

where (see (31), (150) and (151))

$$\boldsymbol{\phi} = \phi_1\boldsymbol{\alpha}_{12} + \phi_2\boldsymbol{\alpha}_2 = \sqrt{2}(\phi_1\mathbf{e}_1 + \phi_2\mathbf{e}_2) \tag{91}$$

where ϕ_1 and ϕ_2 are the corresponding components of the VEV's of the adjoint scalar matrix (6) in given vacuum. In the orthogonal basis of fields a_3 and a_8 they are equal to (see (31))

$$\phi_1 = \frac{1}{2}\left(a_3 + \frac{a_8}{\sqrt{3}}\right), \quad \phi_2 = \frac{1}{2}\left(-a_3 + \frac{a_8}{\sqrt{3}}\right) \tag{92}$$

Let us now show that (89) indeed gives correct tensions for all BPS strings considered before. Start with $r=1$ vacua, from (11) one finds that

$$\boldsymbol{\phi} = -m\left(\mathbf{e}_1 - \frac{\mathbf{e}_2}{2}\right) \tag{93}$$

In $r=1$ vacuum we have only $\mathbf{u}$-string with the charge $\frac{3}{2}\mathbf{u}$, see (64). For this particular string (89) gives

$$T_u = 8\pi\left|\frac{3}{2}\mu m\ \mathbf{u}\left(\mathbf{e}_1 - \frac{\mathbf{e}_2}{2}\right)\right| = 6\pi|\mu m| \tag{94}$$

which coincides with (61). Now consider $r=2$ vacuum, where one has (see (18))

$$\boldsymbol{\phi} = -m_1\mathbf{e}_1 - m_2\mathbf{e}_2 \tag{95}$$

thus, for the (n,k)-string formula (89) gives

$$T_{n,k} = 8\pi|\mu(n\mathbf{e}_1 + k\mathbf{e}_2)(m_1\mathbf{e}_1 + m_2\mathbf{e}_2)| = 4\pi\left|\mu\left(n(2m_1+m_2) + k(2m_2+m_1)\right)\right| \tag{96}$$

and this result coincides with (86). The BPS formula (89) is valid in the limit $\mu \to 0$, $\xi \sim \mu m_A = const$, when one can neglect breaking of $\mathcal{N}=2$ SUSY in effective QED (33) and it assumes also the weak coupling regime $|m_A| \gg \Lambda$.

We conclude this section noting that masses of W-bosons are also determined by the BPS formula, following from (7)

$$m_W = \sqrt{2}|\boldsymbol{\alpha}_W\boldsymbol{\phi}| = 2|\mathbf{q}_W\boldsymbol{\phi}| \tag{97}$$

where $\mathbf{q}_W$ is the (normalized) charge of corresponding W-boson in the Cartan plane. In particular, it means that tensions of $\mathbf{e}_0$-, $\mathbf{e}_1$- and $\mathbf{e}_2$-strings which belong to the adjoint hexagon are proportional to the masses of corresponding W-bosons at least at weak coupling when $m_W \gg \Lambda$. Indeed, charge vectors $\mathbf{q}_s$ of the $\mathbf{e}_0$-, $\mathbf{e}_1$- and $\mathbf{e}_2$-strings coincide with the charges $\mathbf{q}_W$ of related W-bosons, and comparing (89) with (97) we see that tensions of all strings are proportional to the masses of corresponding W-bosons.

[7] We tried to use the string solution in monopole vacuum presented in [15], but unfortunately it does not seem to satisfy all four eqs.(76).

In this section we address the question of existence and stability of (n,k)-strings in $r=2$ vacua studying their interactions at large distances. This allows us to see which strings attract each other and form stable bound states and which do not interact so that their bound state is only marginally stable. First, we consider the interaction of the ANO strings in order to develop necessary technique and then turn to the interactions of (n,k)-strings.

5.1 Interactions of ANO strings

Let us first coalesce the method of effective vertex to calculate the interactions of ANO strings. This method is well-known in instanton physics and is used there to calculate instanton interactions [21, 22, 23, 24]. Here we generalize it to the case of solitonic ANO strings (it also can be easily generalized to any solitonic branes).

Transform the scalar and gauge fields of the ANO string from the "regular" into "singular" gauge making the $U(1)$ gauge transformation with $\exp(-i\vartheta)$, where ϑ is polar angle in the $(1,2)$ plane orthogonal to the string. This gauge transformation is singular at $r=0$ so now the topological charge of the string (flux) comes from small circle around the origin instead of the large circle at $r\to\infty$, and in this gauge the substitution (59) turns into

$$\begin{aligned} \varphi(x) &= \phi(r) \\ A_I(x) &= \frac{\varepsilon_n}{n_e}\epsilon_{IJ}\,\frac{x_J}{r^2}\,f(r) \end{aligned} \tag{98}$$

where the dependence on arbitrary electric charge n_e is restored. Here profile functions $\phi(r)$ and $f(r)$ satisfy the ANO first order equations (cf. with (58))

$$\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}r}\,\phi(r) - f(r)\phi(r) &= 0 \\ -\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r} f(r) + n_e^2 g^2\left(\phi_u^2(r)-\xi\right) &= 0 \end{aligned} \tag{99}$$

and boundary conditions

$$\begin{aligned} \phi(0) = 0, &\quad f(0) = |n| \\ \phi(\infty) = \sqrt{\xi}, &\quad f(\infty) = 0 \end{aligned} \tag{100}$$

were n is integer winding number, while

$$\varepsilon_n = \frac{n}{|n|} = \mathrm{sign}(n) \tag{101}$$

Using equations (99) which determine the exponential fall-off of the functions f and $\phi-\sqrt{\xi}$ at infinity, corresponding to the Yukawa behavior in two transverse dimensions, we get for the large r asymptotic

$$\begin{aligned} \varphi(x) &= \sqrt{\xi}\left(1 - \frac{c_n}{\sqrt{m_\gamma r}}e^{-m_\gamma r} + \cdots\right) \\ A_I(x) &= \frac{\varepsilon_n c_n}{n_e}\epsilon_{IJ}\,\frac{x_J}{r^2}\sqrt{m_\gamma r}\;e^{-m_\gamma r} + \cdots \end{aligned} \tag{102}$$

where c_n is the coefficient to be fixed below, while ε_n is given by (101), both c_n and ε_n depend on the winding number n. We will also need the behavior of the (two-dimensional) dual field strength $\tilde{F} = \frac{1}{2}\epsilon_{IJ}F_{IJ}$ at the infinity in two-dimensional plane $(1,2)$, and from (102) one finds

$$\tilde{F} = \frac{\varepsilon_n c_n m_\gamma^2}{n_e}\,\frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \cdots \tag{103}$$

Now let us work out the effective vertex for the ANO string. This vertex once added to the tree level QED action (e.g. (54)) should reproduce all effects of presence of the ANO string in the framework of perturbation theory (see [21, 22, 23, 24]). For the string with winding number n we propose the following form [8] (up to the overall normalization factor)

$$V_n^{ANO} = \int DX(\boldsymbol{\sigma})\exp\left\{-\int d^2\sigma 2\sqrt{2\pi}c_n\left[\bar{\varphi}\varphi(X(\boldsymbol{\sigma})) + \frac{\varepsilon_n}{2n_e g^2}F_{\mu\nu}(X(\boldsymbol{\sigma}))n_{\mu\nu}\right]\right\} \tag{104}$$

[8] In what follows we are going to keep only bosonic background fields in our expressions for the string interaction, the fermionic terms can be restored by supersymmetry, similar to the instantonic case, see for example [24].

an antisymmetric tensor $n_{\mu\nu}(X(\boldsymbol{\sigma}))$ orthogonal to the string world-sheet at the point $X(\boldsymbol{\sigma})$. For the case of the straight string at rest directed along the third axis $n_{\mu\nu} = 0$ for μ or $\nu = 0, 3$, while $n_{IJ} = \epsilon_{IJ}$ for $\mu, \nu = I, J = 1, 2$. Note, that here we use static parameterization for the string world-sheet, $\sigma_1 = t$ and $\sigma_2 = x_3$, in other words

$$n_{\mu\nu} = \frac{1}{2}\epsilon_{IJ}\frac{\partial X_\alpha}{\partial \sigma_I}\frac{\partial X_\beta}{\partial \sigma_J}\epsilon_{\mu\nu\alpha\beta} \tag{105}$$

and this interaction is nothing but well-known interaction of string with antisymmetric tensor B-field $B = *F$.

To check the expression for the effective vertex (104) let us calculate the correlation function

$$\langle \varphi(x) \ldots \tilde{F}(y) \ldots \rangle_{\text{string}} \tag{106}$$

in the string background assuming that all points x's and y's are far from the axis of the string at $X(\boldsymbol{\sigma})$. On one hand this correlation function can be calculated in semiclassical approximation just substituting classical expressions (102) for the scalar and gauge fields at large distances from the string axis into the correlation function (106), on the other hand the same correlation function can be calculated in perturbation theory as

$$\langle \varphi(x) \ldots \tilde{F}(y) \ldots V_n^{ANO} \rangle \tag{107}$$

once the effective vertex V_n^{ANO} is added to the tree level QED action, and this should give rise to the same result as a substitution of classical fields.

To see this expand scalar field around its VEV $\varphi = \sqrt{\xi} + \delta\varphi$ and write down the bilinear in scalar fields term in the exponential in (104) as

$$\bar{\varphi}\varphi = \xi + \sqrt{\xi}\delta\bar{\varphi} + \sqrt{\xi}\delta\varphi + \delta\bar{\varphi}\delta\varphi + \ldots \tag{108}$$

Consider, first, the linear in quantum fluctuations terms. Expanding the exponential in (107) in $\delta\bar{\varphi}$ and in $\tilde{F}$ we calculate the correlation function (107) to the leading order in coupling constant using the tree level propagator

$$\langle \delta\varphi(x) \int d^2\sigma \overline{\delta\varphi}(X(\boldsymbol{\sigma})) \rangle = \frac{1}{2\pi} K_0(m_\gamma r) \underset{r\to\infty}{\to} \frac{1}{2\sqrt{2\pi}} \frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \ldots \tag{109}$$

for the massive scalar and

$$\langle \tilde{F}(x) \int d^2\sigma \tilde{F}(X(\boldsymbol{\sigma})) \rangle = -\frac{g^2}{2\pi}\frac{\partial^2}{\partial r^2} K_0(m_\gamma r) \underset{r\to\infty}{\to} -\frac{g^2 m_\gamma^2}{2\sqrt{2\pi}} \frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \ldots \tag{110}$$

for the massive vector field, where r is the distance between point x and position of string in $(1, 2)$ plane. These correlation functions are nothing but propagators of two-dimensional theory rewritten in four dimensional notations.

The quadratic in quantum fluctuations term in (108) cannot be verified using just tree level approximation (109), (110), since it is next to leading effect in coupling constant. Still one can restore the quadratic dependence on field φ in the exponential in (104) observing that full effective vertex can depend only on fields $\bar{\varphi}$ and φ rather than upon their VEV's. The result looks a bit surprising from the point of view of string theory, and the origin of this quadratic dependence is that in our picture string tension $T = (2\pi\alpha')^{-1}$ is "dynamical" and determined, in contrast to the fundamental string theory, by condensate of a scalar field.

Let us now determine the constant c_n. Ignoring quantum fluctuations in (104) and substituting the scalar field by its VEV $\sqrt{\xi}$ one gets $2\sqrt{2\pi}c_n\xi$ in the exponential in (104), which should be equal to string tension $2\pi\xi|n|$ (cf. (57)); that gives

$$c_n = \sqrt{\frac{\pi}{2}}|n| \tag{111}$$

Substituting this into (104) we finally obtain

$$V_n^{ANO} = \int DX(\boldsymbol{\sigma}) \exp\left\{-2\pi|n| \int d^2\sigma \left[\bar{\varphi}\varphi(X(\boldsymbol{\sigma})) + \frac{\varepsilon_n}{2n_e g^2} F_{\mu\nu}(X(\boldsymbol{\sigma}))n_{\mu\nu}\right]\right\} \tag{112}$$

and the effective partition function of low-energy QED is now given by

$$Z = \int DA_\mu D\bar{\varphi} D\varphi \exp\left(-S_{QED} - \sum_n V_n^{ANO}\right) \tag{113}$$

in (113) should be treated perturbatively since non-perturbative effects (strings) are already taken into account explicitly.

Let us now use the effective vertex (112) to compute the interaction potential of two straight static strings directed along the third axis at large separation R in $(1,2)$ plane. Expanding $\exp\left(-\sum_n V_n^{ANO}\right)$ in powers of V_n^{ANO} we keep only the term $V_{n_1}^{ANO}V_{n_2}^{ANO}$, related directly to the interaction potential of strings $U_{1,2}$ via

$$\langle V_{n_1}^{ANO}V_{n_2}^{ANO}\rangle = \int DX_1(\boldsymbol{\sigma})DX_2(\boldsymbol{\sigma})e^{-U_{1,2}V_2} \tag{114}$$

where $X_1(\boldsymbol{\sigma})$ and $X_2(\boldsymbol{\sigma})$ correspond to two strings while V_2 is the volume of the two dimensional space in $(0,3)$-plane. Using the propagators (109) and (110) to calculate the correlation function in the l.h.s. of (114) we finally obtain

$$U_{1,2} = -(2\pi)^{\frac{3}{2}}\xi|n_1||n_2|\frac{e^{-m_\gamma R}}{\sqrt{m_\gamma R}}\left[1-\varepsilon_1\varepsilon_2\right] \tag{115}$$

where ε_1, ε_2 refers to the signs of winding numbers (101) of two strings.

We see that potential of string interactions has exponential fall-off at large separations R. The first term in square brackets comes from the exchange by scalar field, and it always gives the attractive contribution to the potential. The second term comes from the photon exchange and its sign depends on the relative signs of n_1 and n_2. If $n_1n_2 > 0$ this term gives repulsion which cancels attraction produced by scalar exchange, which is, of course, a well known result – the BPS ANO strings do not interact. If, however, $n_1n_2 < 0$ (which corresponds to string-antistring interactions) the photon exchange also gives an attraction. Thus the total string-antistring interaction potential is always attractive.

Let us conclude this section presenting the effective vertex of the ANO string which takes into account its motion. It is clear that such generalization of (112) has the form

$$\begin{aligned} V_n^{ANO} &= \int DX(\boldsymbol{\sigma})\exp\left(-S_n^{\text{string}}\right) \\ S_n^{\text{string}} &= 2\pi|n|\int d^2\sigma\left[\sqrt{\det g_{\text{ind}}}\;\bar{\varphi}\varphi(X(\boldsymbol{\sigma})) + \frac{\varepsilon_n}{2n_e g^2}F_{\mu\nu}(X(\boldsymbol{\sigma}))n_{\mu\nu}\right] \end{aligned} \tag{116}$$

Here g_{ind} is the determinant of the induced metric given by

$$g_{IJ}^{\text{ind}} = \frac{\partial X_\mu}{\partial\sigma_I}\frac{\partial X_\mu}{\partial\sigma_J} \tag{117}$$

and interaction with B-field is defined in (105). Besides the usual Nambu-Goto term the string action contains also higher derivative terms omitted in (116). The effective action (116) takes into account interactions of the bulk fields A_μ and φ with the two-dimensional "field" $X_\mu(\boldsymbol{\sigma})$ living on string world sheet.

5.2 Interactions of (n,k)-strings

In this section we apply the effective vertex method to calculate the interaction potential of (n,k)-strings in $r=2$ vacua. First, from the first order equations (76) we get the behavior of the fields $\phi_u(r)$, $\phi_d(r)$ and $f_1(r)$, $f_d(r)$ at infinity $r\to\infty$

$$\begin{aligned} \varphi_u(r) &= \sqrt{\xi}\left(1 - c_{n,k}\frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \cdots\right) \\ \varphi_d(r) &= \sqrt{\omega\xi}\left(1 + c_{n,k}\frac{2}{3(\Omega-\frac{4}{3}\omega)}\frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \cdots\right) \\ \tilde{F}_1(r) &= \varepsilon_{n,k}c_{n,k}\frac{2m_\gamma^2}{\Omega}\frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \cdots \\ \tilde{F}_d(r) &= -\varepsilon_{n,k}c_{n,k}\frac{m_\gamma^2}{\sqrt{3}(\Omega-\frac{4}{3}\omega)}\frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \cdots \end{aligned} \tag{118}$$

where $c_{n,k}$ will be determined below, while sing factors $\varepsilon_{n,k}$ are given by (83). In (118) we use the singular gauge and expressions for dual field strengths $\tilde{F}_1$, $\tilde{F}_d$ related to f_1, f_d via

$$\tilde{F} = -\frac{\varepsilon_{n,k}}{n_e r}\frac{\partial f(r)}{\partial r} \tag{119}$$

Clearly the leading asymptotic behavior in (118) is determined by the lightest gauge and scalar fields so the photon mass in (118) coincides with smaller eigenvalue in formulas (48), (49), i.e. $m_\gamma = (m_\gamma)_-$ and $\Omega = \Omega_-$. In fact (118) already determines the relation between the fields φ_u, φ_d, $\tilde{F}_1$ and $\tilde{F}_d$ and eigenvectors of the mass matrix (45) $\varphi^{(-)}$, $\varphi^{(+)}$, $\tilde{F}^{(-)}$ and $\tilde{F}^{(+)}$

$$\begin{aligned}\delta\varphi_u &= \cos\beta_\varphi \delta\varphi^{(-)} + \sin\beta_\varphi \delta\varphi^{(+)} \\ \delta\varphi_d &= -\sin\beta_\varphi \delta\varphi^{(-)} + \cos\beta_\varphi \delta\varphi^{(+)}\end{aligned} \tag{120}$$

and

$$\begin{aligned}\tilde{F}_1 &= \cos\beta_F \tilde{F}^{(-)} + \sin\beta_F \tilde{F}^{(+)} \\ \tilde{F}_d &= -\sin\beta_F \tilde{F}^{(-)} + \cos\beta_F \tilde{F}^{(+)}\end{aligned} \tag{121}$$

since β_φ and β_F are determined by ratios of the coefficients in front of leading asymptotics in (118)

$$\begin{aligned}\tan\beta_\varphi &= \frac{2}{3}\frac{\sqrt{\omega}}{\Omega - \frac{4}{3}\omega} \\ \tan\beta_F &= \frac{1}{\sqrt{3}}\frac{\Omega}{\Omega - \frac{4}{3}\omega}\end{aligned} \tag{122}$$

Of course, the same relations can be derived directly by diagonalization of the corresponding mass matrices, see sect. 3.3. Using these formulas and asymptotic behavior in (118) for the lightest scalar and vector fields at large r, one gets

$$\begin{aligned}\delta\varphi^{(-)}(r) &= -c_{n,k}\frac{\sqrt{\xi}}{\cos\beta_\varphi}\frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \cdots \\ \tilde{F}^{(-)}(r) &= \varepsilon_{n,k} c_{n,k}\frac{2m_\gamma^2}{\Omega\cos\beta_F}\frac{e^{-m_\gamma r}}{\sqrt{m_\gamma r}} + \cdots\end{aligned} \tag{123}$$

It is clear that the effective vertex for (n,k)-string depends in the leading order only upon the lightest scalar and gauge fields $\varphi^{(-)}$ and $A_\mu^{(-)}$ since only these fields determine the large r behavior of string interaction. Following the same steps which lead us to the effective vertex for the ANO string in sect. 5.1 and using again the propagators (109), (110) we arrive to the following expression for the effective vertex

$$V_{n,k} = \int DX(\boldsymbol{\sigma}) \exp\left\{-\int d^2\sigma 2\sqrt{2\pi} c_{n,k}\left[\frac{\sqrt{\xi}}{\cos\beta_\varphi}\left(\delta\bar{\varphi}^{(-)} + \delta\varphi^{(-)}\right) + \frac{\varepsilon_{n,k}}{g^2\Omega\cos\beta_F}F_{\mu\nu}^{(-)}n_{\mu\nu}\right]\right\} \tag{124}$$

Let us finally fix the constants $c_{n,k}$ here. To do this we rewrite expression (86) for the string tension in the form

$$T_{n,k} = 2\pi\left|n|\varphi_u|^2 + k|\varphi_d|^2\right| \tag{125}$$

which should be consistent with expression in the exponential for $V_{n,k}$. Expanding in (125) the scalar fields around their VEV's we extract the linear term in $\delta\varphi^{(-)}$ using relations (120). Comparing it with the expression linear in $\delta\varphi^{(-)}$ in the exponent in (124) one finds for $c_{n,k}$

$$c_{n,k} = \sqrt{\frac{\pi}{2}}\cos^2\beta_\varphi\left|n + k\tan\beta_\varphi\sqrt{\omega}\right| \tag{126}$$

Substituting this back to (124) and taking into account (125) we finally arrive to the following effective (n,k)-string vertex

$$\begin{aligned}V_{n,k} = \int DX(\boldsymbol{\sigma}) \exp\Big\{-2\pi\int d^2\sigma\Big[&\sqrt{\det g_{\text{ind}}}\left|n|\varphi_u|^2 + k|\varphi_d|^2\right| \\ &+\frac{\varepsilon_{n,k}\cos^2\beta_\varphi}{g^2\Omega\cos\beta_F}\left|n + k\tan\beta_\varphi\sqrt{\omega}\right| F_{\mu\nu}^{(-)}n_{\mu\nu}\Big]\Big\}\end{aligned} \tag{127}$$

where, as in (116) we have already taken into account motion of string.

Now let us turn to the interaction potential of strings at large distances, technically for this purpose it is easier to use effective vertex in the form (124). Expanding $\exp\left(-\sum_{n,k}V_{n,k}\right)$ in powers of $V_{n,k}$ we keep again using tree level propagators (109), (110) leads to the following interaction potential of (n,k)-strings

$$U_{1,2} = -\frac{16}{3}\sqrt{2\pi}\xi c_1 c_2 \frac{(\Omega-\omega)^2 + \frac{1}{3}\omega^2}{\Omega(\Omega - \frac{4}{3}\omega)^2} \frac{e^{-m_\gamma R}}{\sqrt{m_\gamma R}} [1 - \varepsilon_1 \varepsilon_2] \tag{128}$$

where $c_1 \equiv c_{(n_1,k_1)}$ for (n_1,k_1)-string and $c_2 \equiv c_{(n_2,k_2)}$ for (n_2,k_2)-string are given by (126). Quite similar to the case of ANO strings the first term in square brackets here comes from the exchange by two scalars φ_u and φ_d and this interaction always gives rise to attractive potential. The second term arises due to exchange by two photons $A_{1\mu}$ and $A^{(d)}_\mu$, the sign of this contribution is determined by product of the sign factors (83) for two strings. In particular, for $(n_1+\omega k_1)(n_2+\omega k_2) > 0$ the photon exchange gives repulsion which cancels the attraction produced by the scalar exchange and in this case strings do not interact. If instead $(n_1+\omega k_1)(n_2+\omega k_2) < 0$ the interaction potential is attractive so that two strings form a stable bound state.

5.3 Lattice of (n,k)-strings

Now let us use the interaction potential (128) to verify the existence and stability of the (n,k)-string solutions. We are going to determine which strings among the solutions to the BPS first order equations (76) are stable and which strings are non-BPS. The similar approach was used in [25, 26] to study BPS soliton states in two dimensions and various dyon states in Seiberg-Witten theory.

Suppose first, one has two BPS strings (n_1,k_1) and (n_2,k_2) with $(n_1+\omega k_1)(n_2+\omega k_2) > 0$. Then according to (128) these strings do not interact, it means that the bound state of these strings (n_1+n_2, k_1+k_2) exists as a BPS solution to the first order equations (76) but it is only *marginally* stable. The tension of this string is given according to (86) by sum of the tensions of its "components"

$$T_{1\oplus 2} = T_1 + T_2 \tag{129}$$

where $T_s \equiv T_{(n_s,k_s)}$ [9]. If instead $(n_1+\omega k_1)(n_2+\omega k_2) < 0$, two components attract each other and form a stable bound state. However we do not know *a priori* if this bound state is BPS saturated or not, all we know is that its tension is within the bounds

$$|T_1 - T_2| < T_{1\oplus 2} < T_1 + T_2 \tag{130}$$

where the lower bound is given by the BPS formula (86) while the upper bound follows from the potential (128).

From (83) we see that interaction of strings depends on the value of parameter ω. Therefore let us first distinguish physically different regions for this parameter. The masses of W-bosons are classically given by (97) and, as follows from (7), (18) and (47), they are proportional to

$$|Mz|\,,\ \left|\frac{M}{z}\right|,\ \left|M\left(z-\frac{1}{z}\right)\right| \tag{131}$$

where instead of m_1 and m_2 we have introduced new variables M and z defined by

$$\begin{aligned} M^2 &= (2m_1+m_2)(2m_2+m_1) \\ z^2 &= \omega \end{aligned} \tag{132}$$

One can see from (131) that at special values $\omega = 0$, $\omega = 1$ and $\omega = \infty$ one of the W-boson masses vanishes and it corresponds to the restoration of corresponding $SU(2)$ gauge subgroup. However, as we mentioned before this is correct only classically and in quantum theory the $SU(2)$ subgroups are never restored for the theories with $N_f < 4$ (in the next section we consider the theories with four and five flavors where the $SU(2)$ subgroup is restored at $\omega = 1$). Instead the $SU(2)$-subsector runs into strong coupling and one gets monopole and dyon vacua, while W-bosons never become massless [4, 5]. Hence, it is clear that three regions around values $\omega = 0, 1$ and ∞ are in fact strongly coupled and we cannot use our semiclassical analysis there, therefore one has two separated weak coupling regions

$$0 < \omega < 1 \quad \text{and} \quad 1 < \omega < \infty \tag{133}$$

[9] Note that we restrict ourselves only to real and positive values of parameter ω (47). For complex ω one would have far more complicated lattice of (n,k)-strings. In particular the mass formula (86) suggests that we probably can have stable BPS bound states at complex ω which then decay at real ω. Thus the surface $\mathrm{Im}\,\omega = 0$ should be the curve of marginal stability (CMS) for these bound states.

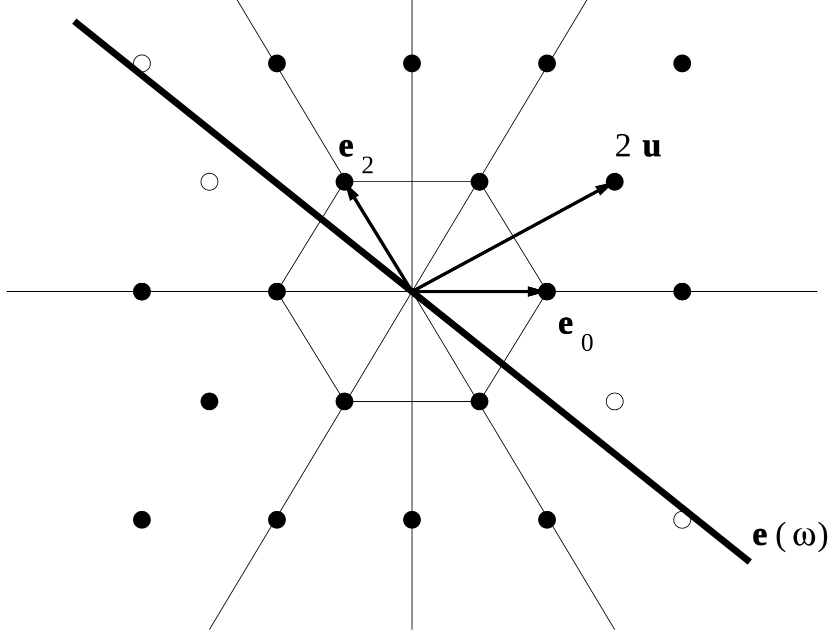

Figure 4: *Lattice of string solutions for $0 < \omega < 1$. The non-BPS strings are drawn by white dots while the BPS strings are depicted by black dots. All BPS strings are marginally stable bound states of "fundamental" $\mathbf{e}_0$- and $\mathbf{e}_2$-strings, corresponding to simple roots of $SU(3)$ algebra. The non-BPS strings can be "crossed" by straight line $n + \omega k = 0$ or $\mathbf{e}(\omega)$ when one moves parameter ω in the region $0 < \omega < 1$.*

and we cannot pass from one region to another within weak coupling regime keeping real ω. Therefore, let us consider these two regions separately.

Start with the region $0 < \omega < 1$ and consider the straight line $n + \omega k = 0$ as is shown on the lattice of (n, k)-strings on fig. 4. This line is directed along the vector

$$\mathbf{e}(\omega) = \omega \mathbf{e}_1 - \mathbf{e}_2 \tag{134}$$

orthogonal to the vector $\boldsymbol{\phi}$, see (95). If charges of two BPS strings are both on the same side out of the line $n + \omega k = 0$ then, according to (128) and (83), these strings do not interact and form marginally stable state. If instead they are from the opposite sides of this line, they attract each other and form a stable bound state.

At $\omega = 0$ the straight line $\mathbf{e}(\omega)$ is directed along $\mathbf{e}_2$ and as we increase ω moves anti-clockwise reaching the vector $\mathbf{e}_0$ at $\omega = 1$, see fig. 4. As the line directed along the vector $\mathbf{e}(\omega)$ hits any knot on the root lattice the BPS formula (86)

$$T^{BPS}_{n,k} = 2\pi\xi |n + \omega k| \tag{135}$$

gives zero for the BPS bound of corresponding string. Thus if within the two $\pi/3$ angles between vectors $-\mathbf{e}_2$ and $\mathbf{e}_0$ ($\mathbf{e}_2$ and $-\mathbf{e}_0$) on fig. 4 (see also fig. 5, where this sector is shown explicitly) all stings were BPS saturated one would get *infinitely many* strings becoming tensionless. To avoid this we have to accept that all these strings are actually *non-BPS* states at $0 < \omega < 1$ (cf. [26]).

Thus, one gets the following picture for the (n, k)-strings at $0 < \omega < 1$: $\mathbf{e}_0$- and $\mathbf{e}_2$-strings ($-\mathbf{e}_0$ and $-\mathbf{e}_2$), with charges proportional to the simple roots $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ of the $SU(3)$ root lattice (see fig. 1) are the lightest BPS stable "elementary" strings in this region. Both strings are from the same side of the line $n + \omega k = 0$ at $0 < \omega < 1$, see fig. 4. Therefore all strings within two $\frac{2\pi}{3}$ angles between vectors $\mathbf{e}_0$ and $\mathbf{e}_2$ ($-\mathbf{e}_0$ and $-\mathbf{e}_2$) labeled by black circles at fig. 4 exist as a BPS solutions of (76) but they are marginally unstable. Instead all strings labeled at fig. 4, by white circles can be considered as bound states of $q\mathbf{e}_0$ and $-p\mathbf{e}_2$ strings (where q, p are integers and $qp > 0$), which are from *different* sides of the line $n + \omega k = 0$. In this case the components are in attractive channel, hence, the composite strings are stable, but as we assumed before they are *not* BPS saturated.

Note as an additional check that $2\mathbf{u} = (2, -1)$ and $(1, 1)$ strings are BPS marginally stable states in our picture at $0 < \omega < 1$. This is in complete agreement with our discussion at the end of sect. 4.2 where we have shown that these strings should exist as a BPS solutions of (76) in some regions around $\omega = 0$ and $\omega = 1$

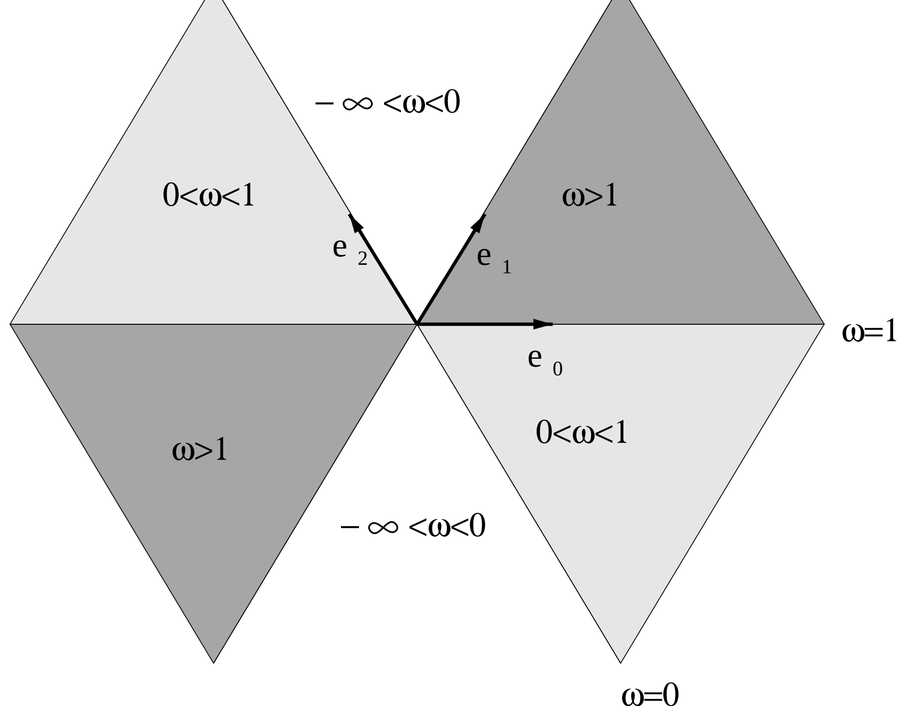

Figure 5: *Different values of the parameter ω. Values $0 < \omega < 1$ are restricted by straight lines $\omega = 0$ along the vector $\mathbf{e}_2$ and $\omega = 1$ along $\mathbf{e}_0$, values $\omega > 1$ are between the lines $\omega = 1$ along $\mathbf{e}_0$ and $\omega = \infty$, which is along the vector $\mathbf{e}_1$.*

respectively. Note also that $\mathbf{e}_2$ and $\mathbf{e}_0$ strings in fact do not become tensionless at $\omega = 0$ and $\omega = 1$ respectively as suggested by BPS formula (135)[10]. As we already mentioned "boundary" values $\omega = 0$ and $\omega = 1$ corresponds to strong coupling where the classical BPS mass formulas should be modified.

Now consider the second region $\omega > 1$, where vector $\mathbf{e}(\omega)$ moves from $\mathbf{e}_0$ at $\omega = 1$ to $\mathbf{e}_1$ at $\omega = \infty$, see fig. 6 and fig. 5. The "elementary" stable BPS strings are now $-\mathbf{e}_0$ and $\mathbf{e}_1$ ($\mathbf{e}_0$ and $-\mathbf{e}_1$). The lattice of BPS marginal strings and non-BPS strings is shown at fig. 6. Note again, that $2\mathbf{d} = (-1, 2)$ and $(1, 1)$-strings exist as BPS solutions in this region exactly as expected from our discussion at the end of sect. 4.2.

Still there is a puzzle to be discussed now. Consider for example $2\mathbf{d} = (-1, 2)$-string, it is marginally stable BPS state at $\omega > 1$ and non-BPS state at $0 < \omega < 1$. This can hardly be correct because, as we already explained in sect. 4.2, the BPS string belong to short multiplets which cannot become long ones under continuous deformation of parameters. The resolution of this puzzle is that once we passed from the region $0 < \omega < 1$ to the region $\omega > 1$ (see fig. 5) we have crossed cuts on the Coulomb branch and due to monodromies quantum numbers of states change [4]. Therefore, say, $2\mathbf{d} = (-1, 2)$-string at $0 < \omega < 1$ and at $\omega > 1$ are, in fact, two *different* states. As we explain below $2\mathbf{u} = (2, -1)$-string at $0 < \omega < 1$ becomes $2\mathbf{d} = (-1, 2)$-string at $\omega > 1$. This string is always marginally stable BPS state.

To see this note that there is a hidden symmetry of a W-boson spectrum (131) [11]

$$\begin{aligned} M &= \text{fixed} \\ z &\to \frac{1}{z} \end{aligned} \tag{136}$$

and let us check now that the spectrum of light particles in $r = 2$ vacuum also respects this symmetry. The eigenvalues of mass matrix are given by eqs. (48), (49), and in terms of variables (132) they get the form

$$(m_\gamma^2)_\pm = \frac{2}{3} g^2 \mu M \left(\frac{1}{z} + z \pm \sqrt{\frac{1}{z^2} - 1 + z^2} \right) \tag{137}$$

[10]See however the next section.

[11]This symmetry is very similar to a symmetry in Toda chain integrable system, which has already appeared in the context of SUSY gauge theories as an elegant way to formulate the Seiberg-Witten exact solution [27].

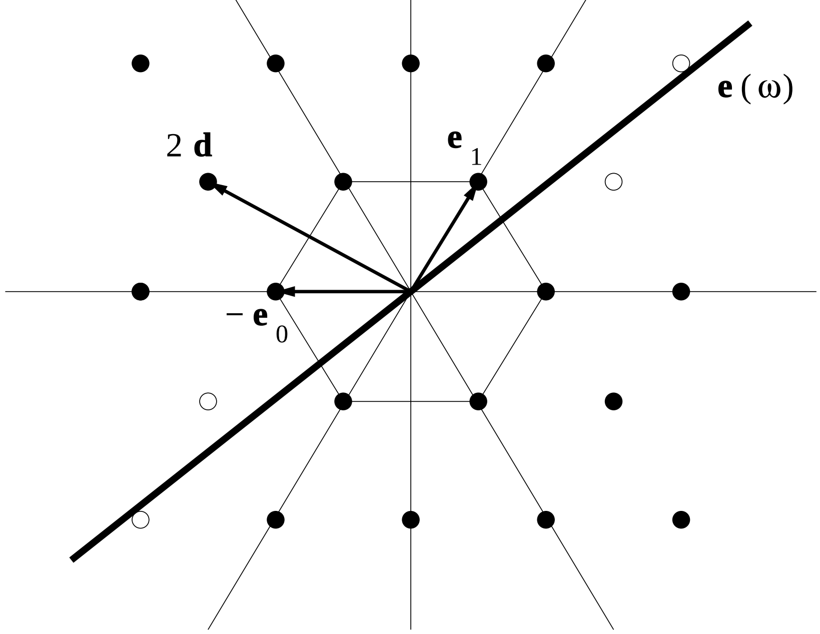

Figure 6: *Lattice of string solutions for $\omega > 1$. Again, the non-BPS strings are switches by white dots while the BPS strings are depicted by black dots. The non-BPS strings can be now "crossed" by straight line $n + \omega k = 0$ or $\mathbf{e}(\omega)$ when one moves parameter ω in the region $1 < \omega < \infty$.*

Therefore we conclude that the low energy spectrum is also invariant under the symmetry (136). Now consider the spectrum of BPS (n, k) strings given by (135), which in terms of M and z (132) reads

$$T^{BPS}_{n,k} = 4\pi\mu M \left| \frac{n}{z} + kz \right| \tag{138}$$

We see that it is indeed invariant under transformation (136) together with exchange in the string quantum numbers

$$(n, k) \leftrightarrow (k, n) \tag{139}$$

Now it is clear that when we pass from the region $0 < \omega < 1$ to the region $\omega > 1$ (see fig. 5) the quantum numbers of strings change according to eq. (139). In particular, $\mathbf{e}_2$-string in the region $0 < \omega < 1$ becomes $\mathbf{e}_1$-string at $\omega > 1$. This string is the stable BPS state. Similarly $2\mathbf{u}$-string in the region $0 < \omega < 1$ turns into $2\mathbf{d}$-string at $\omega > 1$, and this string is always the BPS marginally stable state.

To conclude this section let us sum up the picture of the monopole confinement at $r = 2$ vacuum. The $\mathbf{e}_0$-monopole-antimonopole pair is confined by the $\mathbf{e}_0$-string, while $\mathbf{e}_2$-monopole-antimonopole pair is confined by the $\mathbf{e}_2$-string. The $\mathbf{e}_1$-monopole-antimonopole pair is confined by the $\mathbf{e}_1$-string which is marginally stable bound state of $\mathbf{e}_0$- and $\mathbf{e}_2$-strings. We see that one gets three monopole-antimonopole meson states instead of one. As we have already explained, this "multiplicity" reflects the *Abelian* nature of confinement in Seiberg-Witten theory. For generic values of ω the masses of all three mesons are different (they are determined by different values of string tension).

Finally, let us note, that we predict presence of the non-BPS stable strings which can be considered as bound states of "elementary" BPS strings in attractive channel. These strings can form an "exotic" multi-monopole-multi-antimonopole mesons. Consider, say, the region $0 < \omega < 1$. The example of such string is (-1,2) string which is a stable non-BPS bound state of $(-\mathbf{e}_0)$- and $\mathbf{e}_2$-string. This string bound $\mathbf{e}_2$-monopole and $\mathbf{e}_0$-antimonopole with $\mathbf{e}_2$-antimonopole and $\mathbf{e}_0$-monopole to form an "exotic" meson. Presence of these "exotic" mesons also reflects the Abelian nature of confinement in Seiberg-Witten theory [28, 14]. However, we do not expect such states to appear in theory with non-Abelian confinement.

five flavors

Finally, let us turn to the most physically interesting examples – the $SU(3)$ gauge theory with $N_f = 4$ and especially $N_f = 5$ flavors. As we already explained, the motivation to study such theories is that at certain submanifolds in their moduli space non-Abelian $SU(2)$ gauge symmetry is restored and one can study what happens to confinement and flux tubes in this regime. The reason for restoration of non-Abelian $SU(2)$ gauge symmetry in quantum theory is that at $N_f = 4$ and $N_f = 5$ the corresponding $SU(2)$ subsectors are *not* asymptotically free and they remain to be non-Abelian in weak coupling regime at $m_A \gg \Lambda$ (cf. [9]).

Let us take a closer look at $r = 2$ vacua of these theories. The theory with $N_f = 4$ has six $r = 2$ vacua while the theory with $N_f = 5$ has ten $r = 2$ vacua, see (21). Now consider the point in the parameter space where masses of all flavors m_A become equal. At this point six (ten) vacua of the theory with $N_f = 4$ (5) flavors collide with the two-fold effect. First, as we already discussed in sect. 2.4, the Higgs branch of dimension $8(N_f - 2)$ develops. It touches the Coulomb branch at the point

$$a_3 = 0,\ a_8 = -\sqrt{6}m, \tag{140}$$

where $m = m_A$ is now common value of the quark masses, see (44).

Second, since $a_3 = 0$ the $SU(2)$ gauge subgroup acting in the upper left corner of the adjoint field matrices is classically restored, see (31), [12]. this restoration is also preserved in quantum theory. Namely, the $SU(2)$-subsector for $N_f = 4$ is conformal and the coupling constant does not run, being of the order of

$$g^2 \sim \frac{1}{\log{(m/\Lambda)}} \ll 1 \tag{141}$$

(with $\Lambda \equiv \Lambda_{SU(3)}$; the $SU(3)$ theory with $N_f = 4$ is not conformal), frozen at the scale m where $SU(3)$ group is broken down to $SU(2) \times U(1)$.

For $N_f = 5$ theory the $SU(2)$-subsector has "zero charge" in the infrared limit. The coupling constant is of the order of

$$g^2 \sim -\frac{1}{\log{(\Delta m \Lambda/m^2)}} \ll 1, \tag{142}$$

frozen at lower scale of quark mass difference Δm (note that $\Lambda_{SU(2)} = m^2/\Lambda_{SU(3)}$ for $N_f = 5$).

We consider here special submanifold of the Higgs branch which admits the BPS flux tubes (cf. [7, 18]). We call this submanifold origin of the Higgs branch and it is of dimension $4(N_f - 1)$. One point on this submanifold corresponds to non-zero u-component of the first flavor and non-zero d-component of the second flavor given by

$$\langle \tilde{u}_1 u^1 \rangle = \langle \tilde{d}_2 d^2 \rangle = 3\mu m = \frac{\xi}{2} \tag{143}$$

while all other components are zero. This is a continuation of $r = 2$ vacuum considered in previous sections to equal quark masses, see (19). Other points in the origin of Higgs branch are given by $SU(N_f)$ flavor rotation of (143). The dimension of the origin of the Higgs branch is $4(N_f - 1)$. To see this note that VEV's of u and d-quarks in $r = 2$ vacuum break $SU(N_f)$ symmetry down to $SU(N_f - 2)$. Thus the number of "broken" generators is $4(N_f - 1)$.

Other points on the $8(N_f - 2)$ dimensional Higgs branch correspond to non-zero VEV's of massless moduli fields, and these points do not admit BPS strings. In particular, the ANO strings on Higgs branch were studied in [30], they correspond to limiting case of type I strings with the logarithmically thick tails associated with massless scalar fields. Moreover, the infinitely long strings do not exist [29, 30] and we do not discuss here strings in generic points on Higgs branch..

Now let us focus on the strings arising in some particular vacuum at the origin of Higgs branch. By flavor rotation one can always put this vacuum to the form (143). To study the strings at this vacuum we can apply results of the previous section and take the limit $\omega \to 1$.

Suppose we approach the value $\omega = 1$ from below. Then we have two "elementary" BPS strings, namely $\mathbf{e}_0$ and $\mathbf{e}_2$. All other strings of the string lattice can be obtained as bound states of these two. The BPS formula (86) gives for the $\mathbf{e}_0$ and $\mathbf{e}_2$-string tensions

$$T_0 = 0,\ T_2 = 2\pi\xi \tag{144}$$

[12]Strictly speaking this is true at energies much larger then quark VEV's which break gauge group completely at the scale of order of $\sqrt{\mu m}$, see discussion below.

Let us make a closer look at what happens to this string. In fact, $\mathbf{e}_0$-string corresponds to winding around the $U(1)$ factor associated with the generator λ_3, see (148). This generator belongs to the restored $SU(2)$ subgroup. However, $SU(2)$ group does not admit flux tubes just because $\pi_1(SU(2)) = 0$. Now it is clear that $\mathbf{e}_0$-string becomes unstable at $\omega \to 1$. In fact, the winding just shrinks to zero once the group manifold becomes 3-sphere instead of a circumference.

What happens physically is that $\mathbf{e}_0$-string is broken by $\mathbf{e}_0$-monopole-antimonopole production which does not cost any energy in the limit of equal quark masses. To see this note that mass of the $\mathbf{e}_0$-monopole vanishes in this limit. Namely, the BPS $\mathbf{e}_0$-monopole mass in weak coupling regime has the form [4]

$$M_{e_0} = \sqrt{2}\left|\frac{a_3}{g^2}\right| \tag{145}$$

and vanishes at $a_3 \to 0$.

Thus, we see that although BPS formula (86) suggests that $\mathbf{e}_0$-string becomes tensionless, in fact it becomes unstable and disappears from the spectrum. Hence we are left with only one "elementary" string, namely $\mathbf{e}_2$-string; $\mathbf{e}_1$-string becomes identical to the $\mathbf{e}_2$-string since $\mathbf{e}_1 - \mathbf{e}_2 = \mathbf{e}_0$.

As we already discussed, presence of $N_c - 1$ different strings associated with each $U(1)$ factor of $SU(N_c)$ gauge group broken down to $U(1)^{N_c-1}$ leads to unwanted multiplicity in the hadron spectrum of $\mathcal{N} = 2$ QCD which we do not expect in ordinary QCD [6], and this reflects the $U(1)$ nature of the confinement in Seiberg-Witten theory. In particular, one has generically two "elementary" strings in $SU(3)$ gauge theory. However, in the theory with $N_f = 4$ and $N_f = 5$ flavors with equal masses with the non-Abelian $SU(2)$ subgroup of gauge symmetry restored at $r = 2$ vacuum one of two "elementary" strings becomes unstable and we are left with the *only one* string. Hence, there is no more unwanted multiplicity in the hadron spectrum. Although confinement in this vacuum is due to presence of Abelian $\mathbf{e}_2$-string, the hadron spectrum has multiplicity which we expect in a theory with non-Abelian confinement.

Let us present now more detailed picture of what happens to confined states when we tend to zero the differences of quark masses $\Delta m_{AB} = m_A - m_B$ (assuming that they are all of the same order $\Delta m_{AB} \sim \Delta m$). As we already explained at $\sqrt{\mu m} \ll \Delta m \ll m$ one has weak coupling and the mass of $\mathbf{e}_0$ W-boson is of the order of Δm, while the mass of $\mathbf{e}_0$-monopole is given by (145). When we put Δm well below $\sqrt{\mu m}$ the mass of $\mathbf{e}_0$ W-boson (together with the masses of photons) is determined by complete breaking of $SU(2)$ gauge subgroup by the quark VEV's (instead of adjoint VEV $\langle a_3 \rangle \sim \Delta m$), and it is frozen at the value of order of $\sqrt{\mu m}$. As we reduce Δm the $\mathbf{e}_0$-monopole becomes *lighter* than $\mathbf{e}_0$ W-boson. This shows that eqs.(141), (142) are no longer valid and we enter into strong coupling regime. To understand this, note that presence of simultaneously light quarks (which become massless quark moduli at $\Delta m = 0$) and light monopole means approaching the point of Argyres-Douglas type [31, 32, 33]. Apparently as $\mathbf{e}_0$-monopole becomes lighter and lighter the $\mathbf{e}_0$-string becomes more and more unstable.

In particular, it means that $\mathbf{e}_0$-monopole-antimonopole meson formed by $\mathbf{e}_0$-string with $\mathbf{e}_0$-monopole and antimonopole attached to its ends acquire large width. Eventually at $\Delta m \to 0$ it becomes unobservable as a resonance state and disappear from the spectrum. In contrast, $\mathbf{e}_2$ and $\mathbf{e}_1$-monopoles (they becomes essentially the same states in this limit) are confined by $\mathbf{e}_2$-string. We see that we obtain in fact only one set of monopole-antimonopole meson Regge trajectories as expected in a theory with *non-Abelian* confinement.

Note also that extra multi-monopole-muti-antimonopole "exotic" states formed by stable non-BPS strings (see sect. 5.3) also disappear in the equal mass limit in theories with $N_f = 4$ or $N_f = 5$.

To conclude this section let us make a comment on so called semilocal strings (see [34] for a review). These are string-like solutions which interpolate between BPS strings and two-dimensional sigma model instantons lifted to four dimensions. Instead of ordinary BPS strings these solutions have power fall-off at large r. The semilocal strings arise in the models with additional global symmetry. In the theories with $N_f = 4$ and $N_f = 5$ considered in this section one has global $SU(N_f)$ symmetry at the origin of Higgs branch which leads to presence of massless Goldstone scalars responsible for possible power behavior of string solution at large r. The presence of semilocal string manifests itself as a zero mode of the ordinary BPS string.

However we believe that semilocal strings are not relevant for the problem we study in the paper. First, as is mentioned in sect. 4.1 our strings are BPS only in the leading order in μ/m when perturbation of superpotential (1) reduces to the FI term and $\mathcal{N} = 2$ supersymmetry is not broken. Already in the next to leading order in μ/m these strings turn into type I strings [14]. For type I strings the zero mode becomes the positive mode and semilocal solutions do not arise.

Second, the setup for the problem of confinement we study in this paper is to consider strings of large but tension of order of

$$\xi \frac{\rho^2}{L^2} \tag{146}$$

where ρ is the size of the semilocal string (which is similar to the instanton size). Formula (146) shows that the zero mode associated with size ρ becomes non-zero for the strings of finite length and the minimum of energy corresponds to $\rho = 0$. In this limit semilocal string becomes ordinary BPS string with exponential fall-off of fields at large r.

7 Conclusion

In this paper we have studied the flux tubes in softly broken $\mathcal{N} = 2$ QCD arising in quark vacua of $SU(N_c)$ gauge theory with N_f flavors at weak coupling, in particular focusing on the theory with $SU(3)$ gauge group. These magnetic flux tubes are responsible for the confinement of monopoles. Of course, it would be more desirable to study confinement of quarks arising in dyon vacua at strong coupling via electric flux tubes. Still we believe that the two problems are directly related due to monodromies [4, 5] and it is useful to start with simpler problem of monopole confinement in quark vacua, which one can keep at weak coupling.

Generically $SU(3)$ gauge group is broken down to $U(1)^2$, thus in each $\mathcal{N} = 1$ vacuum two "elementary" flux tubes arise. In $r = 1$ vacua (when only u-quark condense) these are magnetic $\mathbf{u}$-string and electric $\mathbf{e}_2$-string. This hybrid phase arises since classically the gauge group is broken down only to $SU(2) \times U(1)$. However on quantum level the $SU(2)$ factor is further broken down to another $U(1)$ due to the Seiberg-Witten strong coupling mechanism. This results in the condensation of $\mathbf{e}_2$-monopole (dyon) and formation of electric $\mathbf{e}_2$-string. We have shown that strings in $r = 1$ vacua are the standard BPS ANO vortices in the limit of small adjoint masses μ.

In $r = 2$ vacua with non-zero VEV's of u- and d-quarks two "elementary" BPS strings are magnetic $\mathbf{e}_0$ and $\mathbf{e}_2$-string (at the values $0 < \omega < 1$ of VEV's ratio $\omega = \langle \tilde{d}_2 d^2 \rangle / \langle \tilde{u}_1 u^1 \rangle$). These strings are generalization of standard ANO vortices, and they involve two gauge potentials interacting with two scalar fields satisfying BPS first order eqs. (76). We have demonstrated that bound states of multiple $\mathbf{e}_0$-string and multiple $\mathbf{e}_2$-string are marginally stable strings while bound states of multiple $\mathbf{e}_0$-string and multiple $\mathbf{e}_2$-antistring are stable non-BPS strings.

Finally we have considered the theory with $N_f = 4$ and $N_f = 5$ in the limit of equal quark masses. In this limit $SU(2)$ subgroup of the gauge group is not broken even on quantum level in $r = 2$ vacuum. We have shown that in the limit of equal quark masses $\mathbf{e}_0$-string becomes unstable and disappears from the spectrum. The monopole confinement is due to remaining $\mathbf{e}_2$-string which is not distinguishable in this limit from $\mathbf{e}_1$-string. This mechanism eliminates the unwanted multiplicity of the hadron spectrum generically present in Seiberg-Witten theory. The meson spectrum looks as expected in a theory with *non-Abelian* confinement. Namely, one gets the only set of $\mathbf{e}_2$-monopole-antimonopole meson Regge trajectories.

Something peculiar happen to "baryons" in this limit. Of course we cannot really talk about baryons formed by monopoles since monopoles have zero baryon number. Still we can consider neutral baryon-like states formed by $\mathbf{e}_0$, $\mathbf{e}_2$ and $\bar{\mathbf{e}}_1 = -\mathbf{e}_1$ monopoles connected by $\mathbf{e}_0$ and $\mathbf{e}_2$ strings in a chain-like configuration. Generically when $SU(3)$ gauge group is broken down to $U(1)^2$ there are six different "baryon" states of this kind. However, in the limit of equal quark masses when $SU(2)$ gauge subgroup is restored and $\mathbf{e}_0$-string disappears these "baryons" reduce to the only meson formed by $\mathbf{e}_2$-monopole and $\mathbf{e}_2$-antimonopole. It is possible because monopoles have zero baryon number.

This is not exactly what one expects in a theory with non-Abelian confinement. In a theory with non-Abelian confinement one expects to have one set of meson and one set of baryon Regge trajectories. In the theory at hand in the limit of equal quark masses "baryon" states disappear and we are left with only one set of meson trajectories. It would be interesting to study the mechanism of non-Abelian confinement via Abelian flux tubes suggested in this paper for the case of monopole condensation and quark confinement. In particularly this may allow us to find what happens with baryon multiplicity upon non-Abelian gauge subgroup restoration.

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Appendix. SU(3) conventions and beta-functions

The variables ϕ_i are associated with the basis vectors of the fundamental representation **3** (see fig. 1): $\phi_i = \boldsymbol{\phi}\boldsymbol{\mu}_i$, $i = 1, 2, 3$, where $\boldsymbol{\phi}$ is a VEV vector in Cartan plane. Their relation with Cartesian co-ordinates $\boldsymbol{\phi} = (a_3, a_8)$ is given by

$$\langle\Phi\rangle = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} = \frac{a_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{a_8}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \equiv \lambda_3 a_3 + \lambda_8 a_8 \tag{147}$$

i.e. $\langle\Phi\rangle$ is decomposed over well-known diagonal Gell-Mann matrices

$$\lambda_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(\boldsymbol{\alpha}_{12} - \boldsymbol{\alpha}_2) = \frac{1}{2}\boldsymbol{\alpha}_1 = \frac{1}{2}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) \tag{148}$$

and

$$\lambda_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{2\sqrt{3}}(\boldsymbol{\alpha}_{12} + \boldsymbol{\alpha}_2) = \frac{1}{2\sqrt{3}}(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 - 2\boldsymbol{\mu}_3) \tag{149}$$

where we have used the following identification between the vectors in Cartan plane (see fig. 1) and diagonal matrices

$$\boldsymbol{\alpha}_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_3 \tag{150}$$

and

$$\boldsymbol{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3 \tag{151}$$

(vectors (150) and (151) are normalized to $\boldsymbol{\alpha}^2 = 2$ and the factor $\frac{1}{2}$ in (31), (32) is related to conventional normalization of the Gell-Mann matrices $\mathrm{Tr}(\lambda_a\lambda_b) = \frac{1}{2}\delta_{ab}$).

The W-boson masses (7), (97) are proportional to

$$\phi_{ij} \equiv \phi_i - \phi_j = \boldsymbol{\alpha}\boldsymbol{\phi} \tag{152}$$

are given then by the scalar products with the *roots* so that the (perturbative) prepotential for pure gauge theory may be rewritten as

$$\mathcal{F} = \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\Delta_+}(\boldsymbol{\alpha}\boldsymbol{\phi})^2\log\frac{\boldsymbol{\alpha}\boldsymbol{\phi}}{\Lambda} \tag{153}$$

where sum is taken over all positive roots Δ_+. The quadratic part of (153) can be easily rewritten in terms of Cartesian co-ordinates or weights of the fundamental representation, using

$$\boldsymbol{\phi}^2 = \frac{1}{C_V}\sum_{\boldsymbol{\alpha}\in\Delta_+}(\boldsymbol{\alpha}\boldsymbol{\phi})^2 = \sum_{\boldsymbol{\mu}}(\boldsymbol{\mu}\boldsymbol{\phi})^2 \tag{154}$$

or

$$\boldsymbol{\phi}^2 = \frac{1}{N_c}\sum_{i<j}(\phi_{ij})^2 = \frac{1}{2N_c}\sum_{i,j}(\phi_{ij})^2 \underset{\sum_{i=1}^{N_c}\phi_i=0}{=} \sum_{i=1}^{N_c}\phi_i^2 = \frac{1}{2}\sum_{i=1}^{N_c-1}a_i^2 \tag{155}$$

variables $\mathcal{A}_1 = \varphi_1 - \varphi_3 = \boldsymbol{\alpha}_{12}\boldsymbol{\varphi}$ and $\mathcal{A}_2 = \varphi_2 - \varphi_3 = \boldsymbol{\alpha}_2\boldsymbol{\varphi}$, where we used one of the simple roots $\boldsymbol{\alpha}_2$ and the "highest" root $\boldsymbol{\alpha}_{12} = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ (see fig. 1)

$$T_{ij} = \frac{\partial \mathcal{F}}{\partial \mathcal{A}_i \partial \mathcal{A}_j} = \begin{pmatrix} \log \frac{\mathcal{A}_1}{\Lambda} + \log \frac{\mathcal{A}_{12}}{\Lambda} + 3 & -\log \frac{\mathcal{A}_{12}}{\Lambda} \\ -\log \frac{\mathcal{A}_{12}}{\Lambda} & \log \frac{\mathcal{A}_2}{\Lambda} + \log \frac{\mathcal{A}_{12}}{\Lambda} + 3 \end{pmatrix} \tag{156}$$

and it can be rewritten in following co-ordinates

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial a_i \partial a_j} &= \begin{pmatrix} T_{11} + T_{22} - 2T_{12} & \sqrt{3}\,(T_{11} - T_{22}) \\ \sqrt{3}\,(T_{11} - T_{22}) & 3\,(T_{11} + T_{22}) + 6T_{12} \end{pmatrix} = \\ &= \begin{pmatrix} \log \frac{\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_{12}^4}{\Lambda^6} + 6 & \sqrt{3} \log \frac{\mathcal{A}_1}{\mathcal{A}_2} \\ \sqrt{3} \log \frac{\mathcal{A}_1}{\mathcal{A}_2} & 3 \log \frac{\mathcal{A}_1 \mathcal{A}_2}{\Lambda^3} + 18 \end{pmatrix} \end{aligned} \tag{157}$$

If all $\mathcal{A}_i \sim M \to \infty$

$$\frac{\partial \mathcal{F}}{\partial a_i \partial a_j} \sim 6 \log M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(1) \tag{158}$$

with the coefficient $\beta_{YM} = 2N_c = 6$.

One may also consider the contribution of the fundamental multiplets to the set of effective constants (156). They come from the contribution of fundamental massive multiplets to the prepotential

$$\mathcal{F}_f = -\frac{1}{4} \sum_{A=1}^{N_f} \sum_{\boldsymbol{\mu}} (\boldsymbol{\mu\phi} + m_A)^2 \log \frac{\boldsymbol{\mu\phi} + m_A}{\Lambda} \tag{159}$$

whose second derivatives w.r.t. a_i give rise to the contribution to (156) of the form

$$\frac{\partial^2 \mathcal{F}_f}{\partial \mathcal{A}_i \partial \mathcal{A}_j} = \frac{1}{18} \sum_{A=1}^{N_f} \begin{pmatrix} -\log \frac{(B_1^A)^4 B_2^A B^A}{\Lambda^6} & \log \frac{(B_1^A)^2 (B_2^A)^2}{B^A \Lambda^3} \\ \log \frac{(B_1^A)^2 (B_2^A)^2}{B^A \Lambda^3} & -\log \frac{B_1^A (B_2^A)^4 B^A}{\Lambda^6} \end{pmatrix} + const \tag{160}$$

where

$$\begin{aligned} B_1^A &= \boldsymbol{\mu}_1 \boldsymbol{\phi} + m_A = \frac{2}{3}\mathcal{A}_1 - \frac{1}{3}\mathcal{A}_2 + m_A \\ B_2^A &= \boldsymbol{\mu}_2 \boldsymbol{\phi} + m_A = \frac{2}{3}\mathcal{A}_2 - \frac{1}{3}\mathcal{A}_1 + m_A \\ B^A &= -\boldsymbol{\mu}_3 \boldsymbol{\phi} - m_A = \frac{1}{3}\mathcal{A}_1 + \frac{1}{3}\mathcal{A}_2 - m_A \end{aligned} \tag{161}$$

In following co-ordinates one gets

$$\frac{\partial^2 \mathcal{F}_f}{\partial a_i \partial a_j} = \frac{1}{18} \sum_{A=1}^{N_f} \begin{pmatrix} -\log \frac{(B_1^A)^9 (B_2^A)^9}{\Lambda^{18}} & \sqrt{3} \log \frac{(B_2^A)^3}{(B_1^A)^3} \\ \sqrt{3} \log \frac{(B_2^A)^3}{(B_1^A)^3} & -3 \log \frac{B_1^A B_2^A (B^A)^4}{\Lambda^6} \end{pmatrix} + const \tag{162}$$

In the limit $\mathcal{A}_i \sim M \to \infty$ (162) gives rise to

$$\frac{\partial^2 \mathcal{F}_f}{\partial a_i \partial a_j} = -N_f \log M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(1) \tag{163}$$

with the coefficient giving necessary contribution into $\beta_{QCD} = 2N_c - N_f$.

[1] S. Mandelstam, Phys. Rep. **23C** 145 (1976);
A. Polyakov, Nucl. Phys. **B120** (1977) 429;
G. t' Hooft, in Proceed. of the Europ. Phys. Soc. 1975, ed. by A.Zichichi (Editrice Compositori, Bologna, 1976) p. 1225.

[2] A. Abrikosov, Sov. Phys. JETP **32** 1442 (1957).

[3] H. Nielsen and P. Olesen, Nucl. Phys. **B61** 45 (1973).

[4] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19, hep-th/9407087.

[5] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484, hep-th/9408099.

[6] M. Douglas and S. Shenker, Nucl. Phys. **B447** (1995) 271-296, hep-th/9503163.

[7] A. Hanany, M. Strassler and A. Zaffaroni, Nucl. Phys. **B513** (1998) 87, hep-th/9707244.

[8] N. Seiberg, Nucl. Phys. **B435** (1995) 129, hep-th/9411149.

[9] P. Argyres, M. Plesser and N. Seiberg, Nucl. Phys. **B471** (1996) 159-194, hep-th/9603042.

[10] G. Carlino, K. Konishi and H. Murayama, Nucl. Phys. **B590** (2000) 137,hep-th/0005076.

[11] E.Bogomolny, Sov.J.Nucl.Phys. **24** (1976) 449.

[12] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. **344B** (1995) 169; hep-th/ 9411048; hep-th/ 9412158.

[13] P. Argyres and A. Faraggi, Phys. Rev. Lett. **73** (1995) 3931, hep-th/ 9411057.

[14] A. Vainshtein and A. Yung, Nucl. Phys. **B614** (2001) 3, hep-th/0012250.

[15] J. Edelstein, W. Fuertes, J. Mas and J. Guilarte, Phys. Rev. **D62** (2000) 065008, hep-th/0001184.

[16] Z. Hlousek and D. Spector, Nucl.Phys. **B370** (1992) 143;
J. Edelstein, C. Nunez and F. Schaposnik, Phys.Lett. **329B** (1994) 39, hep-th/9311055.

[17] S. Davis, A. Davis, and M. Trodden, Phys. Lett. **B405** (1997) 257, hep-th/9702360.

[18] A. Gorsky and M. Shifman, Phys. Rev. **D61** (2000) 08 5001, hep-th/9909015.

[19] W. Fuertes and J. Guilarte, Phys. Lett. **B437** (1998) 82, hep-th/9807218.

[20] A. Gorsky, A. Vainshtein and A. Yung, Nucl. Phys. **B584** (2000) 197, hep-th/0004087.

[21] C. Callan, R. Dashen and D. Gross, Phys. Rev. **D17** (1978) 2717; **D19** (1979) 1826.

[22] M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. **B165** (1980) 45.

[23] A. Yung, *"Instanton induced Effective Lagrangian in the Gauge-Higgs Theory"*, SISSA 181/90/EP, 1990.

[24] A. Yung, Nucl. Phys. **B485** (1997) 38, hep-th/9605096

[25] A. Ritz, M. Shifman, A. Vainshtein and M. Voloshin, Phys. Rev. **D63** (2001) 065018, hep-th/0006028.

[26] A. Ritz and A. Vainshtein, Nucl. Phys. **B617** (2001) 43, hep-th/0102121

[27] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. **B355** (1995) 466; hep-th/ 9505035.

[28] M. Strassler, Prog. Theor. Phys. Suppl. 131 (1998) 439, hep-th/9803009.

[29] A. Penin, V. Rubakov, P. Tinyakov and S. Troitsky, Phys. Lett. **B389** (1996) 13, hep-th/9609257.

[30] A. Yung, Nucl. Phys. **B562** (1999) 191, hep-th/9906243.

[31] P. Argyres and M. Douglas, Nucl. Phys. **B448** (1995) 93, hep-th/9505062.

[33] A. Yung, *"Confinement near Argyres-Douglas point in $\mathcal{N} = 2$ QCD and low energy version of AdS/CFT correspondence"*, hep-th/0103222.

[34] A. Achucarro and T. Vachaspati, Phys. Rep. **327** (2000) 347-427, hep-ph/9904229.